

# The algebro-geometric solutions for Hunter-Saxton hierarchy

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## Abstract

This paper is dedicated to provide theta function representation of algebro-geometric solutions and related crucial quantities for the Hunter-Saxton (HS) hierarchy through studying a algebro-geometric initial value problem. Our main tools include the polynomial recursive formalism to derive the HS hierarchy, the hyperelliptic curve with finite number of genus, the Baker-Akhiezer functions, the meromorphic function, the Dubrovin-type equations for auxiliary divisors, and the associated trace formulas. With the help of these tools, the explicit representations of the Baker-Akhiezer functions, the meromorphic function, and the algebro-geometric solutions are obtained for the entire HS hierarchy.

## 1 Introduction

The Hunter-Saxton (HS) equation

$$u_{xxt} = -2uu_{xxx} - 4u_xu_{xx}, \quad (1.1)$$

where  $u(x, t)$  is the function of spatial variable  $x$  and time variable  $t$ . It arises in two different physical contexts in two nonequivalent variational forms [1], [2]. The first is shown to describe the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field [1]-[3]. The second is shown to describe the high frequency limit of the Camassa-Holm (CH) equation [5], [6], [32]

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx} \quad (1.2)$$

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which was originally introduced in [5], [6] as model equation for shallow water waves, and obtained independently in [31] with a bi-hamiltonian structure.

The HS equation is a completely integrable system with a bi-hamiltonian structure and hence it possesses a Lax pair, an infinite family of commuting Hamiltonian flows, as well as an associated sequence of conservation laws, (Hunter and Zheng [2], Reyes [20]). The inverse scattering solutions have been obtained by Beals, Sattinger and Szmigielski [19]. Recently, Lenells [23], [24] and also Khesin and Misiołek [22] pointed out that it describes the geodesic flow on the homogeneous space related to the Virasoro group. Bressan and Constantin [25], also Holden [26] constructed a continuous semi-group of weak, dissipative solutions. Yin [27] proved the local existence of strong solutions of the periodic HS equation and showed that all strong solutions-except space independent solutions-blow up in finite time. Gui, Liu and Zhu [28] studied the wave-breaking phenomena and global existence. Furthermore, Morozov [29], Sakovich [30] and Reyes [20], [21] investigated (1.1) from a geometric perspective. However, within the knowledge of the authors, the algebro-geometric solutions of the entire HS hierarchy are not studied yet.

The main task of this paper focuses on the algebro-geometric solutions of the whole HS hierarchy in which (1.1) is just the first member. Algebro-geometric solution, an important feature of integrable system, is a kind of explicit solutions closely related to the inverse spectral theory [4], [7], [9]-[11]. As a degenerated case of the algebro-geometric solution, the multi-soliton solution and periodic solution in elliptic function type may be obtained [7], [8], [33]. A systematic approach, proposed by Gesztesy and Holden to construct algebro-geometric solutions for integrable equations, has been extended to the whole (1+1) dimensional integrable hierarchy, such as the AKNS hierarchy, the CH hierarchy etc. [12]-[15]. Recently, we investigated algebro-geometric solutions for the Gerdjikov-Ivanov hierarchy, the Degasperis-Procesi hierarchy and the modified Camassa-Holm hierarchy [16]-[18].

The outline of the present paper is as follows.

In section 2, based on the polynomial recursion formalism, we derive the HS hierarchy, the associated sequences, and Lax pairs. A hyperelliptic curve  $\mathcal{K}_n$  with arithmetic genus  $n$  is introduced with the help of the characteristic polynomial of Lax matrix  $V_n$  for the stationary HS hierarchy.

In Section 3, we study a meromorphic function  $\phi$  such that  $\phi$  satisfies a nonlinear second-order differential equation. Then we study the properties of the Baker-Akhiezer function  $\psi$ , and furthermore the stationary HS equations

are decomposed into a system of Dubrovin-type equations. The stationary trace formulas are obtained for the HS hierarchy.

In Section 4, we present the first set of our results, the explicit theta function representations of Baker-Akhiezer function, the meromorphic function and the potentials  $u$  for the entire stationary HS hierarchy. Furthermore, we study the initial value problem on an algebro-geometric curve for the stationary HS hierarchy.

In Sections 5 and 6, we extend the analysis in Sections 3 and 4 to the time-dependent case. Each equation in the HS hierarchy is permitted to evolve in terms of an independent time parameter  $t_r$ . As an initial data we use a stationary solution of the  $n$ th equation and then construct a time-dependent solution of the  $r$ th equation in the HS hierarchy. The Baker-Akhiezer function, the meromorphic function, the analogs of the Dubrovin-type equations, the trace formulas, and the theta function representation in Section 4 are all extended to the time-dependent case.

## 2 The HS hierarchy

In this section, we derive the HS hierarchy and the corresponding sequence of zero-curvature pairs by using a polynomial recursion formalism. Moreover, we introduce the hyperelliptic curve connecting to the stationary HS hierarchy.

Throughout this section, let us we make the following hypothesis.

**Hypothesis 2.1** *In the stationary case we assume that*

$$u \in C^\infty(\mathbb{R}), \quad \partial_x^k u \in L^\infty(\mathbb{R}), \quad k \in \mathbb{N}_0. \quad (2.1)$$

*In the time-dependent case we suppose*

$$\begin{aligned} u(\cdot, t) \in C^\infty(\mathbb{R}), \quad \partial_x^k u(\cdot, t) \in L^\infty(\mathbb{R}), \quad k \in \mathbb{N}_0, \quad t \in \mathbb{R}, \\ u(x, \cdot), u_{xx}(x, \cdot) \in C^1(\mathbb{R}), \quad x \in \mathbb{R}. \end{aligned} \quad (2.2)$$

We start by the polynomial recursion formalism. Define  $\{f_l\}_{l \in \mathbb{N}_0}$ ,  $\{g_l\}_{l \in \mathbb{N}_0}$  and  $\{h_l\}_{l \in \mathbb{N}_0}$  recursively by

$$\begin{aligned} f_0 &= 1, \\ f_{l+1,x} &= \mathcal{G}(-4u_{xx}f_{l,x} - 2u_{xxx}f_l), \quad l \in \mathbb{N}_0, \\ g_l &= \frac{1}{2}f_{l+1,x}, \quad l \in \mathbb{N}_0, \\ h_l &= -g_{l+1,x} - u_{xx}f_{l+1}, \quad l \in \mathbb{N}_0, \end{aligned} \quad (2.3)$$

where  $\mathcal{G}$  is given by

$$\begin{aligned}\mathcal{G} : L^\infty(\mathbb{R}) &\rightarrow L^\infty(\mathbb{R}), \\ (\mathcal{G}v)(x) &= \int_{-\infty}^x \int_{-\infty}^\tau v(y) dy d\tau, \quad x \in \mathbb{R}, v \in L^\infty(\mathbb{R}).\end{aligned}\tag{2.4}$$

It is easy to see that  $\mathcal{G}$  is the resolvent of the one-dimensional Laplacian operator, that is

$$\mathcal{G} = \left( \frac{d^2}{dx^2} \right)^{-1}.\tag{2.5}$$

Explicitly, one computes

$$\begin{aligned}f_0 &= 1, \\ f_1 &= -2u + c_1, \\ f_2 &= \mathcal{G}(4uu_{xx} + 2u_x^2) - c_1 2u + c_2, \\ g_0 &= -u_x, \\ g_1 &= \frac{1}{2}\mathcal{G}(8u_x u_{xx} + 4u u_{xxx}) - c_1 u_x, \\ h_0 &= -\frac{1}{2}f_{2,xx} - u_{xx}f_1,\end{aligned}\tag{2.6}$$

where  $\{c_l\}_{l \in \mathbb{N}_0} \subset \mathbb{C}$  are integration constants and we have used the assumption

$$f_l(u)|_{u=0} = c_l, \quad g_l(u)|_{u=0} = c_l, \quad h_l(u)|_{u=0} = c_l, \quad l \in \mathbb{N}.\tag{2.7}$$

Next we introduce the corresponding homogeneous coefficients  $\hat{f}_l, \hat{g}_l$ , and  $\hat{h}_l$ , defined through taking  $c_k = 0$  for  $k = 1, \dots, l$ ,

$$\begin{aligned}\hat{f}_0 &= f_0 = 1, \quad \hat{f}_l = f_l|_{c_k=0, k=1, \dots, l}, \quad l \in \mathbb{N}, \\ \hat{g}_0 &= g_0 = -u_x, \quad \hat{g}_l = g_l|_{c_k=0, k=1, \dots, l}, \quad l \in \mathbb{N}, \\ \hat{h}_0 &= h_0, \quad \hat{h}_l = h_l|_{c_k=0, k=1, \dots, l}, \quad l \in \mathbb{N}.\end{aligned}\tag{2.8}$$

Hence one can easily conclude that

$$f_l = \sum_{k=0}^l c_{l-k} \hat{f}_k, \quad g_l = \sum_{k=0}^l c_{l-k} \hat{g}_k, \quad h_l = \sum_{k=0}^l c_{l-k} \hat{h}_k, \quad l \in \mathbb{N}_0,\tag{2.9}$$

with

$$c_0 = 1.\tag{2.10}$$

Now we consider the following  $2 \times 2$  matrix isospectral problem

$$\psi_x = U(u, z)\psi = \begin{pmatrix} 0 & 1 \\ -z^{-1}u_{xx} & 0 \end{pmatrix} \psi \quad (2.11)$$

and an auxiliary problem

$$\psi_{t_n} = V_n(z)\psi, \quad (2.12)$$

where  $V_n(z)$  is defined by

$$V_n(z) = \begin{pmatrix} -G_n(z) & F_{n+1}(z) \\ z^{-1}H_n(z) & G_n(z) \end{pmatrix} \quad z \in \mathbb{C} \setminus \{0\}, \quad n \in \mathbb{N}_0, \quad (2.13)$$

assuming  $F_{n+1}$ ,  $G_n$  and  $H_n$  to be polynomials of degree  $n$  with  $C^\infty$  coefficients with respect to  $x$ . The compatibility condition between (2.11) and (2.12) yields the stationary zero-curvature equation

$$-V_{n,x} + [U, V_n] = 0, \quad (2.14)$$

that is

$$F_{n+1,x} = 2G_n, \quad (2.15)$$

$$H_{n,x} = 2u_{xx}G_n, \quad (2.16)$$

$$zG_{n,x} = -H_n - u_{xx}F_{n+1}. \quad (2.17)$$

From (2.15)-(2.17), a direct calculation shows that

$$\frac{d}{dx} \det(V_n(z, x)) = -\frac{1}{z^2} \frac{d}{dx} \left( z^2 G_n(z, x)^2 + z F_{n+1}(z, x) H_n(z, x) \right) = 0 \quad (2.18)$$

and hence  $z^2 G_n^2 + z F_{n+1} H_n$  is  $x$ -independent implying

$$z^2 G_n^2 + z F_{n+1} H_n = R_{2n+2}, \quad (2.19)$$

where the integration constant  $R_{2n+2}$  is a polynomial of degree  $2n+2$  with respect to  $z$ . If  $\{E_m\}_{m=0, \dots, 2n+1}$  denote its zeros, then

$$R_{2n+2}(z) = (u_x^2 + h_0) \prod_{m=0}^{2n+1} (z - E_m), \quad E_0 = 0, \quad \{E_m\}_{m=1, \dots, 2n+1} \in \mathbb{C}. \quad (2.20)$$

Here we must emphasize that the coefficient  $(u_x^2 + h_0)$  is a constant. In fact, (2.18) equals

$$2z^2 G_n G_{n,x} + z F_{n+1} H_{n,x} + z H_n F_{n+1,x} = 0. \quad (2.21)$$

Comparing the coefficients of powers  $z^{2n+2}$  yields

$$2g_0g_{0,x} + f_0h_{0,x} + h_0f_{0,x} = 0, \quad (2.22)$$

which together with (2.6) we obtain

$$2u_xu_{xx} + h_{0,x} = 0. \quad (2.23)$$

Hence

$$u_x^2 + h_0 = \partial^{-1}(2u_xu_{xx} + h_{0,x}) = \text{Constant}. \quad (2.24)$$

For simplicity, we denote it by  $a^2$ . Then  $R_{2n+2}(z)$  can be rewritten as

$$R_{2n+2}(z) = a^2 \prod_{m=0}^{2n+1} (z - E_m), \quad E_0 = 0, \quad \{E_m\}_{m=1, \dots, 2n+1} \in \mathbb{C}. \quad (2.25)$$

In order to derive the corresponding hyperelliptic curve, we compute the characteristic polynomial  $\det(yI - zV_n)$  of Lax matrix  $zV_n$ ,

$$\begin{aligned} \det(yI - zV_n) &= y^2 - z^2G_n(z)^2 - F_{n+1}(z)H_n(z) \\ &= y^2 - R_{2n+2}(z) = 0. \end{aligned} \quad (2.26)$$

Equation (2.26) naturally leads to the hyperelliptic curve  $\mathcal{K}_n$ , where

$$\mathcal{K}_n : \mathcal{F}_n(z, y) = y^2 - R_{2n+2}(z) = 0. \quad (2.27)$$

The stationary zero-curvature equation (2.14) implies polynomial recursion relations (2.3). Introducing the following polynomial  $F_{n+1}(z)$ ,  $G_n(z)$  and  $H_n(z)$  with respect to the spectral parameter  $z$ ,

$$F_{n+1}(z) = \sum_{l=0}^{n+1} f_l z^{n+1-l}, \quad (2.28)$$

$$G_n(z) = \sum_{l=0}^n g_l z^{n-l}, \quad (2.29)$$

$$H_n(z) = \sum_{l=0}^n h_l z^{n-l}. \quad (2.30)$$

Inserting (2.28)-(2.30) into (2.15)-(2.17) then yields the recursions relations (2.3) for  $f_l$ ,  $l = 0, \dots, n+1$ , and  $g_l$ ,  $l = 0, \dots, n$ . For fixed  $n \in \mathbb{N}_0$ , by using (2.17), we obtain the recursion for  $h_l$ ,  $l = 0, \dots, n-1$  in (2.3) and

$$h_n = -u_{xx}f_{n+1}. \quad (2.31)$$

Moreover, from (2.16), one infers that

$$h_{n,x} - 2u_{xx}g_n = 0, \quad n \in \mathbb{N}_0. \quad (2.32)$$

Hence, insertion of the equation (2.31) and

$$f_{n+1,x} - 2g_n = 0 \quad (2.33)$$

into (2.32), we derive the stationary HS hierarchy,

$$\text{s-HS}_n(u) = u_{xxx}f_{n+1}(u) + 2u_{xx}f_{n+1,x}(u) = 0, \quad n \in \mathbb{N}_0. \quad (2.34)$$

Explicitly, the first few equations are as follows

$$\begin{aligned} \text{s-HS}_0(u) &= -2uu_{xxx} - 4u_xu_{xx} + c_1u_{xxx} = 0, \\ \text{s-HS}_1(u) &= u_{xxx}\mathcal{G}(4uu_{xx} + 2u_x^2) + 2u_{xx}\mathcal{G}(8u_xu_{xx} + 4uu_{xxx}) \\ &\quad + c_1(-2uu_{xxx} - 4u_xu_{xx}) + c_2u_{xxx} = 0, \\ &\text{etc.} \end{aligned} \quad (2.35)$$

By definition, the set of solutions of (2.34) represents the class of algebro-geometric HS solutions, with  $n$  ranging in  $\mathbb{N}_0$  and  $c_l$  in  $\mathbb{C}$ ,  $l \in \mathbb{N}$ . We call the stationary algebro-geometric HS solutions  $u$  as HS potentials at times.

**Remark 2.2** *Here we emphasize that if  $u$  satisfies one of the stationary HS equations in (2.34), then it must satisfy infinitely many such equations of order higher than  $n$  for certain choices of integration constants  $c_l$ , this is a common characteristic of the general integrable soliton equations such as the KdV, AKNS and CH hierarchies [15].*

Next, we introduce the corresponding homogeneous polynomials  $\hat{F}_{l+1}, \hat{G}_l, \hat{H}_l$  defined by

$$\hat{F}_{l+1}(z) = F_{l+1}(z)|_{c_k=0, k=1,\dots,l} = \sum_{k=0}^{l+1} \hat{f}_k z^{l+1-k}, \quad l = 0, \dots, n, \quad (2.36)$$

$$\hat{G}_l(z) = G_l(z)|_{c_k=0, k=1,\dots,l} = \sum_{k=0}^l \hat{g}_k z^{l-k}, \quad l = 0, \dots, n, \quad (2.37)$$

$$\hat{H}_l(z) = H_l(z)|_{c_k=0, k=1,\dots,l} = \sum_{k=0}^l \hat{h}_k z^{l-k}, \quad l = 0, \dots, n-1, \quad (2.38)$$

$$\hat{H}_n(z) = -u_{xx}\hat{f}_{n+1} + \sum_{k=0}^{n-1} \hat{h}_k z^{n-k}. \quad (2.39)$$

Then the corresponding homogeneous formalism of (2.34) are given by

$$\widehat{\text{s-HS}}_n(u) = \text{s-HS}_n(u)|_{c_l=0, l=1,\dots,n} = 0, \quad n \in \mathbb{N}_0. \quad (2.40)$$

We will end this section by introducing the time-dependent HS hierarchy. This means that  $u$  are now considered as functions of both space and time. We introduce a deformation parameter  $t_n \in \mathbb{R}$  in  $u$ , replacing  $u(x)$  by  $u(x, t_n)$ , for each equation in the hierarchy. In addition, we note that the definitions (2.11), (2.13) and (2.28)-(2.30) of  $U$ ,  $V_n$  and  $F_{n+1}, G_n$  and  $H_n$  are still apply. Then the compatibility condition yields the zero-curvature equation

$$U_{t_n} - V_{n,x} + [U, V_n] = 0, \quad n \in \mathbb{N}_0, \quad (2.41)$$

namely

$$-u_{xxt_n} - H_{n,x} + 2u_{xx}G_n = 0, \quad (2.42)$$

$$F_{n+1,x} = 2G_n, \quad (2.43)$$

$$zG_{n,x} = -H_n - u_{xx}F_{n+1}. \quad (2.44)$$

For fixed  $n \in \mathbb{N}$ , insertion of the polynomial expressions for  $F_{n+1}$ ,  $G_n$  and  $H_n$  into (2.42)-(2.44), respectively, then we derive the relations (2.3) for  $f_l|_{l=0,\dots,n+1}$ ,  $g_l|_{l=0,\dots,n}$ ,  $h_l|_{l=0,\dots,n-1}$  and

$$h_n = -u_{xx}f_{n+1}. \quad (2.45)$$

Moreover, from (2.42), we infer that

$$-u_{xxt_n} - h_{n,x} + 2u_{xx}g_n = 0, \quad n \in \mathbb{N}_0. \quad (2.46)$$

Hence, together (2.45) and

$$f_{n+1,x} = 2g_n, \quad (2.47)$$

(2.46) admits the time-dependent HS hierarchy,

$$\begin{aligned} \text{HS}_n(u) &= -u_{xxt_n} + u_{xxx}f_{n+1}(u) + 2u_{xx}f_{n+1,x}(u) = 0, \\ &\quad (x, t_n) \in \mathbb{R}^2, \quad n \in \mathbb{N}_0. \end{aligned} \quad (2.48)$$

Explicitly, the first few equations are as follows

$$\begin{aligned} \text{HS}_0(u) &= -u_{xxt_0} - 2uu_{xxx} - 4u_{xx}u_x + c_1u_{xxx} = 0, \\ \text{HS}_1(u) &= -u_{xxt_1} + u_{xxx}\mathcal{G}(4uu_{xx} + 2u_x^2) + 2u_{xx}\mathcal{G}(8u_xu_{xx} + 4uu_{xxx}) \\ &\quad + c_1(-2uu_{xxx} - 4u_xu_{xx}) + c_2u_{xxx} = 0, \\ &\text{etc.} \end{aligned} \quad (2.49)$$



The first equation  $\text{HS}_0(u) = 0$  (with  $c_1 = 0$ ) in the hierarchy represents the Hunter-Saxton equation discussed in section 1. Similarly, one can introduce the corresponding homogeneous HS hierarchy by

$$\widehat{\text{HS}}_n(u) = \text{HS}_n(u)|_{c_l=0, l=1,\dots,n} = 0, \quad n \in \mathbb{N}_0. \quad (2.50)$$

In fact, since the Lenard recursion formalism is almost universally adopted in the contemporary literature on the integrable soliton equations, it might be worthwhile to adopt Gesztesy method, an alternative approach using the polynomial recursion relations.

### 3 The stationary HS formalism

In this section we focus our attention on the stationary case. By using the polynomial recursion formalism described in section 2, we define a fundamental meromorphic function  $\phi(P, x)$  on a hyperelliptic curve  $\mathcal{K}_n$ . Moreover, we study the properties of the Baker-Akhiezer function  $\psi(P, x, x_0)$ , Dubrovin-type equations and trace formulas.

We emphasize that the analysis about the stationary case described in section 2 also holds here for the present context.

The hyperelliptic curve  $\mathcal{K}_n$

$$\begin{aligned} \mathcal{K}_n : \mathcal{F}_n(z, y) &= y^2 - R_{2n+2}(z) = 0, \\ R_{2n+2}(z) &= a^2 \prod_{m=0}^{2n+1} (z - E_m), \quad E_0 = 0, \quad \{E_m\}_{m=1,\dots,2n+1} \in \mathbb{C}, \end{aligned} \quad (3.1)$$

which is compactified by joining two points at infinity,  $P_{\infty\pm}$ ,  $P_{\infty+} \neq P_{\infty-}$ , but for notational simplicity the compactification is also denoted by  $\mathcal{K}_n$ . Points  $P$  on

$$\mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}\}$$

are represented as pairs  $P = (z, y(P))$ , where  $y(\cdot)$  is the meromorphic function on  $\mathcal{K}_n$  satisfying

$$\mathcal{F}_n(z, y(P)) = 0.$$

The complex structure on  $\mathcal{K}_n$  is defined in the usual way by introducing local coordinates

$$\zeta_{Q_0} : P \rightarrow (z - z_0)$$

near points  $Q_0 = (z_0, y(Q_0)) \in \mathcal{K}_n \setminus P_0$ ,  $P_0 = (0, 0)$ , which are neither branch nor singular points of  $\mathcal{K}_n$ ; near  $P_0$ , the local coordinates are

$$\zeta_{P_0} : P \rightarrow z^{1/2};$$

near the points  $P_{\infty\pm} \in \mathcal{K}_n$ , the local coordinates are

$$\zeta_{P_{\infty\pm}} : P \rightarrow z^{-1},$$

and similarly at branch and singular points of  $\mathcal{K}_n$ . Hence,  $\mathcal{K}_n$  becomes a two-sheeted hyperelliptic Riemann surface of genus  $n \in \mathbb{N}_0$  (possibly with a singular affine part) in a standard manner.

We also notice that fixing the zeros  $E_0 = 0, E_1, \dots, E_{2n+1}$  of  $R_{2n+2}$  discussed in (3.1) leads to the curve  $\mathcal{K}_n$  is fixed. Then the integration constants  $c_1, \dots, c_n$  in  $f_n$  are uniquely determined, which is the symmetric functions of  $E_1, \dots, E_{2n+1}$ .

The holomorphic map  $*$ , changing sheets, is defined by

$$\begin{aligned} * : \begin{cases} \mathcal{K}_n \rightarrow \mathcal{K}_n, \\ P = (z, y_j(z)) \rightarrow P^* = (z, y_{j+1 \pmod{2}}(z)), \quad j = 0, 1, \\ P^{**} := (P^*)^*, \quad \text{etc.}, \end{cases} \end{aligned} \quad (3.2)$$

where  $y_j(z)$ ,  $j = 0, 1$  denote the two branches of  $y(P)$  satisfying  $\mathcal{F}_n(z, y) = 0$ . Finally, positive divisors on  $\mathcal{K}_n$  of degree  $n$  are denoted by

$$\mathcal{D}_{P_1, \dots, P_n} : \begin{cases} \mathcal{K}_n \rightarrow \mathbb{N}_0, \\ P \rightarrow \mathcal{D}_{P_1, \dots, P_n} = \begin{cases} k & \text{if } P \text{ occurs } k \text{ times in } \{P_1, \dots, P_n\}, \\ 0 & \text{if } P \notin \{P_1, \dots, P_n\}. \end{cases} \end{cases} \quad (3.3)$$

Next, we define the stationary Baker-Akhiezer function  $\psi(P, x, x_0)$  on  $\mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0\}$  by

$$\begin{aligned} \psi(P, x, x_0) &= \begin{pmatrix} \psi_1(P, x, x_0) \\ \psi_2(P, x, x_0) \end{pmatrix}, \\ \psi_x(P, x, x_0) &= U(u(x), z(P))\psi(P, x, x_0), \\ zV_n(u(x), z(P))\psi(P, x, x_0) &= y(P)\psi(P, x, x_0), \\ \psi_1(P, x_0, x_0) &= 1; \\ P = (z, y) &\in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0\}, \quad (x, x_0) \in \mathbb{R}^2. \end{aligned} \quad (3.4)$$

Closely related to  $\psi(P, x, x_0)$  is the following meromorphic function  $\phi(P, x)$  on  $\mathcal{K}_n$  defined by

$$\phi(P, x) = z \frac{\psi_{1,x}(P, x, x_0)}{\psi_1(P, x, x_0)}, \quad P \in \mathcal{K}_n, \quad x \in \mathbb{R} \quad (3.5)$$

with

$$\psi_1(P, x, x_0) = \exp \left( z^{-1} \int_{x_0}^x \phi(P, x') dx' \right), \quad P \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0\}. \quad (3.6)$$

Then, based on (3.4) and (3.5), a direct calculation shows that

$$\begin{aligned} \phi(P, x) &= \frac{y + zG_n(z, x)}{F_{n+1}(z, x)} \\ &= \frac{zH_n(z, x)}{y - zG_n(z, x)}, \end{aligned} \quad (3.7)$$

and

$$\psi_2(P, x, x_0) = \psi_1(P, x, x_0)\phi(P, x)/z. \quad (3.8)$$

We note that  $F_{n+1}$  and  $H_n$  are polynomials with respect to  $z$  of degree  $n+1$  and  $n$ , respectively. Hence we may write

$$F_{n+1}(z) = \prod_{j=0}^n (z - \mu_j), \quad H_n(z) = h_0 \prod_{l=1}^n (z - \nu_l). \quad (3.9)$$

Moreover, defining

$$\hat{\mu}_j(x) = (\mu_j(x), -\mu_j(x)G_n(\mu_j(x), x)) \in \mathcal{K}_n, \quad j = 0, \dots, n, \quad x \in \mathbb{R}, \quad (3.10)$$

and

$$\hat{\nu}_l(x) = (\nu_l(x), \nu_l(x)G_n(\nu_l(x), x)) \in \mathcal{K}_n, \quad l = 1, \dots, n, \quad x \in \mathbb{R}. \quad (3.11)$$

Due to assumption (2.1),  $u$  is smooth and bounded, and hence  $F_{n+1}(z, x)$  and  $H_n(z, x)$  share the same property. Thus, we infer that

$$\mu_j, \nu_l \in C(\mathbb{R}), \quad j = 0, \dots, n, \quad l = 1, \dots, n. \quad (3.12)$$

here  $\mu_j, \nu_l$  may have appropriate multiplicities.

The branch of  $y(\cdot)$  near  $P_{\infty\pm}$  is fixed according to

$$\lim_{\substack{|z(P)| \rightarrow \infty \\ P \rightarrow P_{\infty\pm}}} \frac{y(P)}{z(P)G_n(z(P), x)} = \mp 1. \quad (3.13)$$

Also by (3.7), the divisor  $(\phi(P, x))$  of  $\phi(P, x)$  is given by

$$(\phi(P, x)) = \mathcal{D}_{P_0\hat{\nu}(x)}(P) - \mathcal{D}_{\hat{\mu}_0(x)\hat{\mu}(x)}(P). \quad (3.14)$$

That means,  $P_0, \hat{\nu}_1(x), \dots, \hat{\nu}_n(x)$  are the  $n+1$  zeros of  $\phi(P, x)$  and  $\hat{\mu}_0(x), \hat{\mu}_1(x), \dots, \hat{\mu}_n(x)$  are its  $n+1$  poles. These zeros and poles can be abbreviated in the following form

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}, \quad \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \text{Sym}^n(\mathcal{K}_n). \quad (3.15)$$

Let us recall the holomorphic map (3.2),

$$\begin{aligned} * : \begin{cases} \mathcal{K}_n \rightarrow \mathcal{K}_n, \\ P = (z, y_j(z)) \rightarrow P^* = (z, y_{j+1(\text{mod } 2)}(z)), \quad j = 0, 1, \\ P^{**} := (P^*)^*, \quad \text{etc.}, \end{cases} \end{aligned} \quad (3.16)$$

where  $y_j(z)$ ,  $j = 0, 1$  satisfy  $\mathcal{F}_n(z, y) = 0$ , namely

$$(y - y_0(z))(y - y_1(z)) = y^2 - R_{2n+2}(z) = 0. \quad (3.17)$$

Hence from (3.17), we can easily get

$$\begin{aligned} y_0 + y_1 &= 0, \\ y_0 y_1 &= -R_{2n+2}(z), \\ y_0^2 + y_1^2 &= 2R_{2n+2}(z). \end{aligned} \quad (3.18)$$

Further properties of  $\phi(P, x)$  are summarized as follows.

**Lemma 3.1** *Under the assumption (2.1), let  $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0\}$ , and  $x \in \mathbb{R}$ , and  $u$  satisfies the  $n$ th stationary HS equation (2.34). Then*

$$\phi_x(P) + z^{-1}\phi(P)^2 = -u_{xx}, \quad (3.19)$$

$$\phi(P)\phi(P^*) = -\frac{zH_n(z)}{F_{n+1}(z)}, \quad (3.20)$$

$$\phi(P) + \phi(P^*) = \frac{2zG_n(z)}{F_{n+1}(z)}, \quad (3.21)$$

$$\phi(P) - \phi(P^*) = \frac{2y}{F_{n+1}(z)}. \quad (3.22)$$

**Proof.** A direct calculation shows that (3.19) holds. Let us now prove (3.20)-(3.22). Without loss of generality, let  $y_0(P) = y(P)$ . From (3.7),

(2.19) and (3.18), we arrive at

$$\begin{aligned}
\phi(P)\phi(P^*) &= \frac{y_0 + zG_n}{F_{n+1}} \times \frac{y_1 + zG_n}{F_{n+1}} \\
&= \frac{y_0y_1 + (y_0 + y_1)zG_n + z^2G_n^2}{F_{n+1}^2} \\
&= \frac{-R_{2n+2} + z^2G_n^2}{F_{n+1}^2} = z \frac{-F_{n+1}H_n}{F_{n+1}^2} \\
&= -\frac{zH_n}{F_{n+1}}, \tag{3.23}
\end{aligned}$$

$$\begin{aligned}
\phi(P) + \phi(P^*) &= \frac{y_0 + zG_n}{F_{n+1}} + \frac{y_1 + zG_n}{F_{n+1}} \\
&= \frac{(y_0 + y_1) + 2zG_n}{F_{n+1}} = \frac{2zG_n}{F_{n+1}}, \tag{3.24}
\end{aligned}$$

$$\begin{aligned}
\phi(P) - \phi(P^*) &= \frac{y_0 + zG_n}{F_{n+1}} - \frac{y_1 + zG_n}{F_{n+1}} \\
&= \frac{(y_0 - y_1)}{F_{n+1}} = \frac{2y_0}{F_{n+1}} = \frac{2y}{F_{n+1}}. \tag{3.25}
\end{aligned}$$

Hence we complete the proof.  $\square$

Let us detail the properties of  $\psi(P, x, x_0)$  below.

**Lemma 3.2** *Under the assumption (2.1), let  $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0\}$ ,  $(x, x_0) \in \mathbb{R}^2$ , and  $u$  satisfies the  $n$ th stationary HS equation (2.34). Then*

$$\psi_1(P, x, x_0) = \left( \frac{F_{n+1}(z, x)}{F_{n+1}(z, x_0)} \right)^{1/2} \exp\left( \frac{y}{z} \int_{x_0}^x F_{n+1}(z, x')^{-1} dx' \right), \tag{3.26}$$

$$\psi_1(P, x, x_0)\psi_1(P^*, x, x_0) = \frac{F_{n+1}(z, x)}{F_{n+1}(z, x_0)}, \tag{3.27}$$

$$\psi_2(P, x, x_0)\psi_2(P^*, x, x_0) = -\frac{H_n(z, x)}{zF_{n+1}(z, x_0)}, \tag{3.28}$$

$$\psi_1(P, x, x_0)\psi_2(P^*, x, x_0) + \psi_1(P^*, x, x_0)\psi_2(P, x, x_0) = 2\frac{G_n(z, x)}{F_{n+1}(z, x_0)}, \tag{3.29}$$

$$\psi_1(P, x, x_0)\psi_2(P^*, x, x_0) - \psi_1(P^*, x, x_0)\psi_2(P, x, x_0) = \frac{-2y}{zF_{n+1}(z, x_0)}. \tag{3.30}$$

**Proof.** Equation (3.26) can be proven through the following procedure. Using (2.15), the expression of  $\psi_1$ , (3.6) and (3.7), we obtain

$$\begin{aligned}\psi_1(P, x, x_0) &= \exp \left( z^{-1} \int_{x_0}^x \frac{y + zG_n(z, x')}{F_{n+1}(z, x')} dx' \right) \\ &= \exp \left( z^{-1} \int_{x_0}^x \left( \frac{y}{F_{n+1}(z, x')} + \frac{1}{2} \frac{F_{n+1, x'}(z, x')}{F_{n+1}(z, x')} \right) dx' \right),\end{aligned}\tag{3.31}$$

which implies (3.26). Moreover, (3.6) and (3.8) together with (3.20)-(3.22) yields

$$\begin{aligned}\psi_1(P, x, x_0)\psi_1(P^*, x, x_0) &= \exp \left( z^{-1} \int_{x_0}^x (\phi(P) + \phi(P^*)) dx' \right) \\ &= \exp \left( z^{-1} \int_{x_0}^x \frac{2zG_n(z, x')}{F_{n+1}(z, x')} dx' \right) \\ &= \exp \left( \int_{x_0}^x \frac{F_{n+1, x'}(z, x')}{F_{n+1}(z, x')} dx' \right) \\ &= \frac{F_{n+1}(z, x)}{F_{n+1}(z, x_0)},\end{aligned}\tag{3.32}$$

$$\begin{aligned}\psi_2(P, x, x_0)\psi_2(P^*, x, x_0) &= z^{-2}\psi_1(P, x, x_0)\phi(P, x)\psi_1(P^*, x, x_0)\phi(P^*, x) \\ &= z^{-2} \frac{F_{n+1}(z, x)}{F_{n+1}(z, x_0)} \frac{(-zH_n(z, x))}{F_{n+1}(z, x)} \\ &= -\frac{H_n(z, x)}{zF_{n+1}(z, x_0)},\end{aligned}\tag{3.33}$$

$$\begin{aligned}\psi_1(P, x, x_0)\psi_2(P^*, x, x_0) + \psi_1(P^*, x, x_0)\psi_2(P, x, x_0) &= \psi_1(P)\psi_1(P^*)\phi(P^*)/z + \psi_1(P^*)\psi_1(P)\phi(P)/z \\ &= \psi_1(P)\psi_1(P^*)(\phi(P) + \phi(P^*))/z \\ &= \frac{F_{n+1}(z, x)}{F_{n+1}(z, x_0)} \frac{2zG_n(z, x)}{zF_{n+1}(z, x)} \\ &= \frac{2G_n(z, x)}{F_{n+1}(z, x_0)},\end{aligned}\tag{3.34}$$

$$\begin{aligned}\psi_1(P, x, x_0)\psi_2(P^*, x, x_0) - \psi_1(P^*, x, x_0)\psi_2(P, x, x_0) &= \psi_1(P)\psi_1(P^*)\phi(P^*)/z + \psi_1(P^*)\psi_1(P)\phi(P)/z \\ &= \psi_1(P)\psi_1(P^*)(\phi(P^*) - \phi(P))/z\end{aligned}$$

$$\begin{aligned}
&= \frac{F_{n+1}(z, x)}{F_{n+1}(z, x_0)} \frac{-2y}{zF_{n+1}(z, x)} \\
&= \frac{-2y}{zF_{n+1}(z, x_0)}.
\end{aligned} \tag{3.35}$$

Hence (3.27)-(3.30) hold.  $\square$

In Lemma 3.2 if we choose

$$\psi_1(P) = \psi_{1,+}, \quad \psi_1(P^*) = \psi_{1,-}, \quad \psi_2(P) = \psi_{2,+}, \quad \psi_2(P^*) = \psi_{2,-},$$

then (3.27)-(3.30) imply

$$(\psi_{1,+}\psi_{2,-} - \psi_{1,-}\psi_{2,+})^2 = (\psi_{1,+}\psi_{2,-} + \psi_{1,-}\psi_{2,+})^2 - 4\psi_{1,+}\psi_{2,-}\psi_{1,-}\psi_{2,+}, \tag{3.36}$$

which is equivalent to the basic identity (2.19),  $z^2 G_n^2 + zF_{n+1}H_n = R_{2n+2}$ . This fact reveals the relations between our approach and the algebro-geometric solutions of the HS hierarchy.

**Remark 3.3** *The definition of stationary Baker-Akhiezer function  $\psi$  of the HS hierarchy is analogous to that in the context of KdV or AKNS hierarchies. But the crucial difference is that  $P_0$  is a essential singularity of  $\psi$  in the HS hierarchy, which is the same as in CH hierarchy, but different from the KdV or AKNS hierarchy. This fact will be showed in the asymptotic expansions of  $\psi$  in next section.*

Furthermore, we derive Dubrovin-type equations, which are first-order coupled systems of differential equations and govern the dynamics of the zeros  $\mu_j(x)$  and  $\nu_l(x)$  of  $F_{n+1}(z, x)$  and  $H_n(z, x)$  with respect to  $x$ . We recall the affine part of  $\mathcal{K}_n$  is nonsingular if

$$\begin{aligned}
E_0 &= 0, \quad \{E_m\}_{m=1, \dots, 2n+1} \subset \mathbb{C} \setminus \{0\}, \\
E_m &\neq E_{m'} \quad \text{for } m \neq m', m, m' = 1, \dots, 2n+1.
\end{aligned} \tag{3.37}$$

**Lemma 3.4** *Assume that (2.1) holds and  $u$  satisfies the  $n$ th stationary HS equation (2.34).*

(i) *If the zeros  $\{\mu_j(x)\}_{j=0, \dots, n}$  of  $F_{n+1}(z, x)$  remain distinct for  $x \in \Omega_\mu$ , where  $\Omega_\mu \subseteq \mathbb{R}$  is an open interval, then  $\{\mu_j(x)\}_{j=0, \dots, n}$  satisfy the system of differential equations,*

$$\mu_{j,x} = 2 \frac{y(\hat{\mu}_j)}{\mu_j} \prod_{\substack{k=0 \\ k \neq j}}^n (\mu_j(x) - \mu_k(x))^{-1}, \quad j = 0, \dots, n, \tag{3.38}$$

with initial conditions

$$\{\hat{\mu}_j(x_0)\}_{j=0,\dots,n} \in \mathcal{K}_n, \quad (3.39)$$

for some fixed  $x_0 \in \Omega_\mu$ . The initial value problems (3.38), (3.39) have a unique solution satisfying

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 0, \dots, n. \quad (3.40)$$

(ii) If the zeros  $\{\nu_l(x)\}_{l=1,\dots,n}$  of  $H_n(z, x)$  remain distinct for  $x \in \Omega_\nu$ , where  $\Omega_\nu \subseteq \mathbb{R}$  is an open interval, then  $\{\nu_l(x)\}_{l=1,\dots,n}$  satisfy the system of differential equations,

$$\nu_{l,x} = -2 \frac{u_{xx}}{h_0} \frac{y(\hat{\nu}_l)}{\nu_l} \prod_{\substack{k=1 \\ k \neq l}}^n (\nu_l(x) - \nu_k(x))^{-1}, \quad l = 1, \dots, n, \quad (3.41)$$

with initial conditions

$$\{\hat{\nu}_l(x_0)\}_{l=1,\dots,n} \in \mathcal{K}_n, \quad (3.42)$$

for some fixed  $x_0 \in \Omega_\nu$ . The initial value problems (3.41), (3.42) have a unique solution satisfying

$$\hat{\nu}_l \in C^\infty(\Omega_\nu, \mathcal{K}_n), \quad l = 1, \dots, n. \quad (3.43)$$

**Proof.** For our convenience, let us focus on (3.38) and (3.40), the proof of (3.41) and (3.43) follows in an identical manner. The derivatives of (3.9) with respect to  $x$  take on

$$F_{n+1,x}(\mu_j) = -\mu_{j,x} \prod_{\substack{k=0 \\ k \neq j}}^n (\mu_j(x) - \mu_k(x)). \quad (3.44)$$

On the other hand, inserting  $z = \mu_j$  into equation (2.15) leads to

$$F_{n+1,x}(\mu_j) = 2G_n(\mu_j) = 2 \frac{y(\hat{\mu}_j)}{-\mu_j}. \quad (3.45)$$

Comparing (3.44) with (3.45) gives (3.38). The proof of smoothness assertion (3.40) is analogous to the mCH case in our latest paper [18].  $\square$

Let us now turn to the trace formulas of the HS invariants, which is the expressions of  $f_l$  and  $h_l$  in terms of symmetric functions of the zeros  $\mu_j$  and  $\nu_l$  of  $F_{n+1}$  and  $H_n$ , respectively. Here, we just consider the simplest case.



**Lemma 3.5** *If (2.1) holds and  $u$  satisfies the  $n$ th stationary HS equation (2.34), then*

$$u = \frac{1}{2} \sum_{j=0}^n \mu_j - \frac{1}{2} \sum_{m=0}^{2n+1} E_m. \quad (3.46)$$

**Proof.** By comparison of the coefficient of  $z^n$  of  $F_{n+1}$  in (2.28) and (3.9), taking account into (2.6) yields

$$-2u + c_1 = - \sum_{j=0}^n \mu_j. \quad (3.47)$$

The constant  $c_1$  can be determined by a long straightforward calculation comparing the coefficients of  $z^{2n+1}$  in (2.19), which leads to

$$c_1 = - \sum_{m=0}^{2n+1} E_m. \quad (3.48)$$

## 4 Stationary algebro-geometric solutions of HS hierarchy

In this section we continue our study of the stationary HS hierarchy, and will obtain explicit Riemann theta function representations for the meromorphic function  $\phi$ , and especially, for the potentials  $u$  of the stationary HS hierarchy.

Let us begin with the asymptotic properties of  $\phi$  and  $\psi_j, j = 1, 2$ .

**Lemma 4.1** *Assume that (2.1) to hold and  $u$  satisfies the  $n$ th stationary HS equation (2.34). Moreover, let  $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0\}$ ,  $(x, x_0) \in \mathbb{R}^2$ . Then*

$$\phi(P) \underset{\zeta \rightarrow 0}{=} -u_x + O(\zeta), \quad P \rightarrow P_{\infty\pm}, \quad \zeta = z^{-1}, \quad (4.1)$$

$$\phi(P) \underset{\zeta \rightarrow 0}{=} i a \left( \prod_{m=1}^{2n+1} E_m \right)^{1/2} f_{n+1}^{-1} \zeta + O(\zeta^2), \quad P \rightarrow P_0, \quad \zeta = z^{1/2}, \quad (4.2)$$

and

$$\psi_1(P, x, x_0) \underset{\zeta \rightarrow 0}{=} \exp\left((u(x_0) - u(x))\zeta + O(\zeta^2)\right), \quad (4.3)$$

$$P \rightarrow P_{\infty\pm}, \quad \zeta = z^{-1},$$

$$\psi_2(P, x, x_0) \underset{\zeta \rightarrow 0}{=} O(\zeta) \exp\left((u(x_0) - u(x))\zeta + O(\zeta^2)\right), \quad (4.4)$$

$$P \rightarrow P_{\infty\pm}, \quad \zeta = z^{-1},$$

$$\begin{aligned}\psi_1(P, x, x_0) &\underset{\zeta \rightarrow 0}{=} \exp\left(\frac{i}{\zeta} \int_{x_0}^x dx' a\left(\prod_{m=1}^{2n+1} E_m\right)^{1/2} f_{n+1}(x')^{-1} + O(1)\right), \\ P &\rightarrow P_0, \quad \zeta = z^{1/2},\end{aligned}\tag{4.5}$$

$$\begin{aligned}\psi_2(P, x, x_0) &\underset{\zeta \rightarrow 0}{=} O(\zeta^{-1}) \exp\left(\frac{i}{\zeta} \int_{x_0}^x dx' a\left(\prod_{m=1}^{2n+1} E_m\right)^{1/2} f_{n+1}(x')^{-1} + O(1)\right), \\ P &\rightarrow P_0, \quad \zeta = z^{1/2}.\end{aligned}\tag{4.6}$$

**Proof.** Under the local coordinates  $\zeta = z^{-1}$  near  $P_{\infty\pm}$  and  $\zeta = z^{1/2}$  near  $P_0$ , the existence of the asymptotic expansions of  $\phi$  is clear from its explicit expressions in (3.7). Next, we use the Riccati-type equation (3.19) to compute the explicit expansion coefficients. Inserting the ansatz

$$\phi \underset{z \rightarrow \infty}{=} \phi_0 + \phi_1 z^{-1} + O(z^{-2})\tag{4.7}$$

into (3.19) and comparing the powers of  $z^0$  then yields (4.1). Similarly, inserting the ansatz

$$\phi \underset{z \rightarrow 0}{=} \phi_1 z^{1/2} + \phi_2 z + O(z^{3/2})\tag{4.8}$$

into (3.19) and comparing the power of  $z^0$  then yields (4.2), where we used (2.31) and

$$f_{n+1} h_n = -a^2 \prod_{m=1}^{2n+1} E_m,\tag{4.9}$$

which can be obtained by (2.19). Finally, expansions (4.3)-(4.6) follow up by (3.6), (3.8), (4.1) and (4.2).  $\square$

**Remark 4.2** From (4.5) and (4.6), we note the unusual fact that  $P_0$  is the essential singularity of  $\psi_j$ ,  $j = 1, 2$ , this is consistent with Remark 3.3. Also the leading-order exponential term  $\psi_j$ ,  $j = 1, 2$ , near  $P_0$  is  $x$ -dependent, which makes matters worse. This is in sharp contrast to standard Baker-Akhiezer functions that typically feature a linear behavior with respect to  $x$  such as  $\exp(c(x - x_0)\zeta^{-1})$  near  $P_0$ .

Let us now introduce the holomorphic differentials  $\eta_l(P)$  on  $\mathcal{K}_n$

$$\eta_l(P) = \frac{a}{y(P)} z^{l-1} dz, \quad l = 1, \dots, n,\tag{4.10}$$

and choose a homology basis  $\{a_j, b_j\}_{j=1}^n$  on  $\mathcal{K}_n$  in such a way that the intersection matrix of the cycles satisfies

$$a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j, k = 1, \dots, n.$$

Define an invertible matrix  $E \in GL(n, \mathbb{C})$  as follows

$$\begin{aligned} E &= (E_{j,k})_{n \times n}, \quad E_{j,k} = \int_{a_k} \eta_j, \\ \underline{c}(k) &= (c_1(k), \dots, c_n(k)), \quad c_j(k) = (E^{-1})_{j,k}, \end{aligned} \quad (4.11)$$

and the normalized holomorphic differentials

$$\omega_j = \sum_{l=1}^n c_j(l) \eta_l, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad \int_{b_k} \omega_j = \tau_{j,k}, \quad j, k = 1, \dots, n. \quad (4.12)$$

Apparently, the matrix  $\tau$  is symmetric and has a positive-definite imaginary part.

The symmetric function  $\Phi_n^{(j)}(\bar{\mu})$  and  $\Psi_{n+1}(\bar{\mu})$  are defined by

$$\Phi_n^{(j)}(\bar{\mu}) = (-1)^n \prod_{\substack{p=0 \\ p \neq j}}^n \mu_p, \quad (4.13)$$

$$\Psi_{n+1}(\bar{\mu}) = (-1)^{n+1} \prod_{p=0}^n \mu_p. \quad (4.14)$$

The following result shows that the nonlinearity of the Abel map in the HS hierarchy. This feature is analogous to CH hierarchy but sharp apposed to other integrable soliton equations such as KdV and AKNS hierarchies.

**Theorem 4.3** *Assume (2.1) to hold and suppose that  $\{\hat{\mu}_j\}_{j=0, \dots, n}$  satisfies the stationary Dubrovin equations (3.38) on  $\Omega_\mu$  and remain distinct for  $x \in \Omega_\mu$ , where  $\Omega_\mu \subseteq \mathbb{R}$  is an open interval. Introducing the associated divisor  $\mathcal{D}_{\hat{\mu}_0(x)\hat{\underline{\mu}}(x)}$ . Then*

$$\partial_x \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x)\hat{\underline{\mu}}(x)}) = -\frac{2a}{\Psi_{n+1}(\bar{\mu}(x))} \underline{c}(1), \quad x \in \Omega_\mu. \quad (4.15)$$

*In particular, the Abel map does not linearize the divisor  $\mathcal{D}_{\hat{\mu}_0(x)\hat{\underline{\mu}}(x)}$  on  $\Omega_\mu$ , where  $\bar{\mu}(x) = (\mu_0(x), \mu_1(x), \dots, \mu_n(x)) = \mu_0(x)\underline{\mu}(x)$ .*

**Proof.** Is easy to see that

$$\frac{1}{\mu_j} = \frac{\prod_{p=0, p \neq j}^n \mu_p}{\prod_{p=0}^n \mu_p} = -\frac{\Phi_n^{(j)}(\bar{\mu})}{\Psi_{n+1}(\bar{\mu})}, \quad j = 1, \dots, n. \quad (4.16)$$

Let

$$\underline{\omega} = (\omega_1, \dots, \omega_n), \quad (4.17)$$

and choose a appropriate base point  $Q_0$ . Then we arrive at

$$\begin{aligned} \partial_x \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x)} \underline{\hat{\mu}}(x)) &= \partial_x \left( \sum_{j=0}^n \int_{Q_0}^{\hat{\mu}_j} \underline{\omega} \right) = \sum_{j=0}^n \mu_{j,x} \sum_{k=1}^n \underline{c}(k) \frac{a \mu_j^{k-1}}{y(\hat{\mu}_j)} \\ &= \sum_{j=0}^n \sum_{k=1}^n \frac{2a \mu_j^{k-1}}{\mu_j} \frac{1}{\prod_{\substack{l=0 \\ l \neq j}}^n (\mu_j - \mu_l)} \underline{c}(k) \\ &= -\frac{2a}{\Psi_{n+1}(\bar{\mu})} \sum_{j=0}^n \sum_{k=1}^n \underline{c}(k) \frac{\mu_j^{k-1}}{\prod_{\substack{l=0 \\ l \neq j}}^n (\mu_j - \mu_l)} \Phi_n^{(j)}(\bar{\mu}) \\ &= -\frac{2a}{\Psi_{n+1}(\bar{\mu})} \sum_{j=0}^n \sum_{k=1}^n \underline{c}(k) (U_{n+1}(\bar{\mu}))_{k,j} (U_{n+1}(\bar{\mu}))_{j,1}^{-1} \\ &= -\frac{2a}{\Psi_{n+1}(\bar{\mu})} \sum_{k=1}^n \underline{c}(k) \delta_{k,1} \\ &= -\frac{2a}{\Psi_{n+1}(\bar{\mu})} \underline{c}(1), \end{aligned} \quad (4.18)$$

where we used

$$(U_{n+1}(\bar{\mu})) = \left( \frac{\mu_j^{k-1}}{\prod_{\substack{l=0 \\ l \neq j}}^n (\mu_j - \mu_l)} \right)_{j=0}^n, \quad (U_{n+1}(\bar{\mu}))^{-1} = \left( \Phi_n^{(j)}(\bar{\mu}) \right)_{j=0}^n, \quad (4.19)$$

the definition of which is analogous to (E.25) and (E.26) in [15].  $\square$

The analogous results hold for the corresponding divisor  $\mathcal{D}_{\hat{\nu}(x)}$  associated with  $\phi(P, x)$  can be obtained in the same way.

Next, we introduce <sup>1</sup>

$$\begin{aligned}
\hat{\underline{B}}_{Q_0} : \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}\} &\rightarrow \mathbb{C}^n, \\
P &\mapsto \hat{\underline{B}}_{Q_0}(P) = (\hat{B}_{Q_0,1}, \dots, \hat{B}_{Q_0,n}) \\
&= \begin{cases} \int_{Q_0}^P \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)}, & n = 1 \\ \left( \int_{Q_0}^P \eta_2, \dots, \int_{Q_0}^P \eta_n, \int_{Q_0}^P \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)} \right), & n \geq 2, \end{cases}
\end{aligned} \tag{4.20}$$

where

$$\tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)} = \frac{a z^n}{y(P)} dz$$

denotes a differential of the third kind with simple poles at  $P_{\infty+}$  and  $P_{\infty-}$  and corresponding residues  $+1$  and  $-1$ , respectively. Moreover,

$$\begin{aligned}
\hat{\underline{\beta}}_{Q_0} : \text{Sym}^n(\mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}\}) &\rightarrow \mathbb{C}^n, \\
\mathcal{D}_{\underline{Q}} &\mapsto \hat{\underline{\beta}}_{Q_0}(\mathcal{D}_{\underline{Q}}) = \sum_{j=1}^n \hat{\underline{B}}_{Q_0}(Q_j), \\
\underline{Q} = \{Q_1, \dots, Q_n\} &\in \text{Sym}^n(\mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}\}).
\end{aligned} \tag{4.21}$$

The following result is a special case of Theorem 4.3, which will be used to provide the proper change of variables to linear the divisor  $\mathcal{D}_{\hat{\mu}_0(x)\hat{\underline{\mu}}(x)}$  associated with  $\phi(P, x)$ .

**Theorem 4.4** *Assume that (2.1) holds and the statements of  $\mu_j$  in Theorem 4.3 are all true. Then*

$$\partial_x \sum_{j=0}^n \int_{Q_0}^{\hat{\mu}_j(x)} \eta_1 = -\frac{2a}{\Psi_{n+1}(\bar{\mu}(x))}, \quad x \in \Omega_{\mu}, \tag{4.22}$$

$$\partial_x \hat{\underline{\beta}}(\mathcal{D}_{\hat{\underline{\mu}}(x)}) = \begin{cases} 2a, & n = 1, \\ 2a (0, \dots, 0, 1), & n \geq 2, \end{cases} \quad x \in \Omega_{\mu}. \tag{4.23}$$

**Proof.** Equations (4.22) is a special case (4.15) and (4.23) follows from (4.18). Alternatively, one can follow the same way as shown in Theorem 4.3 to derive (4.22) and (4.23).  $\square$

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<sup>1</sup> Here we choose the same path of integration from  $Q_0$  and  $P$  in all integrals in (4.20) and (4.21).

Let  $\theta(\underline{z})$  denote the Riemann theta function associated with  $\mathcal{K}_n$  and an appropriately fixed homology basis. We assume  $\mathcal{K}_n$  to be nonsingular. Next, choosing a convenient base point  $Q_0 \in \mathcal{K}_n \setminus \{\hat{\mu}_0(x), P_0\}$ , the vector of Riemann constants  $\Xi_{Q_0}$  is given by (A.66) [15], and the Abel maps  $\underline{A}_{Q_0}(\cdot)$  and  $\underline{\alpha}_{Q_0}(\cdot)$  are defined by

$$\begin{aligned} \underline{A}_{Q_0} : \mathcal{K}_n &\rightarrow J(\mathcal{K}_n) = \mathbb{C}^n / L_n, \\ P &\mapsto \underline{A}_{Q_0}(P) = (\underline{A}_{Q_0,1}(P), \dots, \underline{A}_{Q_0,n}(P)) = \left( \int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_n \right) \pmod{L_n} \end{aligned} \quad (4.24)$$

and

$$\begin{aligned} \underline{\alpha}_{Q_0} : \text{Div}(\mathcal{K}_n) &\rightarrow J(\mathcal{K}_n), \\ \mathcal{D} &\mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_n} \mathcal{D}(P) \underline{A}_{Q_0}(P), \end{aligned} \quad (4.25)$$

where  $L_n = \{\underline{z} \in \mathbb{C}^n \mid \underline{z} = \underline{N} + \tau \underline{M}, \underline{N}, \underline{M} \in \mathbb{Z}^n\}$ .

Let

$$\omega_{\hat{\mu}_0(x)P_0}^{(3)}(P) = \frac{a}{y} \prod_{j=1}^n (z - \lambda_j) dz \quad (4.26)$$

be the normalized differential of the third kind holomorphic on  $\mathcal{K}_n \setminus \{\hat{\mu}_0(x), P_0\}$  with simple poles at  $\hat{\mu}_0(x)$  and  $P_0$  with residues  $\pm 1$ , respectively, that is,

$$\begin{aligned} \omega_{\hat{\mu}_0(x)P_0}^{(3)}(P) &\underset{\zeta \rightarrow 0}{=} (\zeta^{-1} + O(1)) d\zeta, \quad \text{as } P \rightarrow \hat{\mu}_0(x), \\ \omega_{\hat{\mu}_0(x)P_0}^{(3)}(P) &\underset{\zeta \rightarrow 0}{=} (-\zeta^{-1} + O(1)) d\zeta, \quad \text{as } P \rightarrow P_0, \end{aligned} \quad (4.27)$$

where the local coordinate are given by

$$\zeta = z^{-1} \quad \text{for } P \text{ near } \hat{\mu}_0(x), \quad \zeta = z^{1/2} \quad \text{for } P \text{ near } P_0, \quad (4.28)$$

and the constants  $\{\lambda_j\}_{j=1,\dots,n}$  are determined by the normalization condition

$$\int_{a_k} \omega_{\hat{\mu}_0(x)P_0}^{(3)} = 0, \quad k = 1, \dots, n.$$

Then

$$\int_{Q_0}^P \omega_{\hat{\mu}_0(x)P_0}^{(3)}(P) \underset{\zeta \rightarrow 0}{=} \ln \zeta + e_0 + O(\zeta), \quad \text{as } P \rightarrow \hat{\mu}_0(x), \quad (4.29)$$

$$\int_{Q_0}^P \omega_{\hat{\mu}_0(x)P_0}^{(3)}(P) \underset{\zeta \rightarrow 0}{=} -\ln \zeta + d_0 + O(\zeta), \quad \text{as } P \rightarrow P_0, \quad (4.30)$$

for some constants  $e_0, d_0 \in \mathbb{C}$  that arise from the integrals at their lower limits  $Q_0$ . We also note that

$$\underline{A}_{Q_0}(P) - \underline{A}_{Q_0}(P_{\infty \pm}) \underset{\zeta \rightarrow 0}{=} \pm \underline{U} \zeta + O(\zeta^2), \quad \text{as } P \rightarrow P_{\infty \pm}, \quad \underline{U} = \underline{c}(n). \quad (4.31)$$

The following abbreviations are used for our convenience:

$$\begin{aligned} \underline{z}(P, \underline{Q}) &= \Xi_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \\ P &\in \mathcal{K}_n, \quad \underline{Q} = (Q_1, \dots, Q_n) \in \text{Sym}^n(\mathcal{K}_n), \end{aligned} \quad (4.32)$$

where  $\underline{z}(\cdot, \underline{Q})$  is independent of the choice of base point  $Q_0$ .

Moreover, from Theorem 4.3 and Theorem 4.4 we note that the Abel map dose not linearize the divisor  $\mathcal{D}_{\hat{\mu}_0(x)\hat{\mu}(x)}$ . However, the change of variables

$$x \mapsto \tilde{x} = \int^x dx' \left( \frac{2a}{\Psi_{n+1}(\bar{\mu}(x'))} \right) \quad (4.33)$$

linearizes the Abel map  $\underline{A}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(\tilde{x})\hat{\mu}(\tilde{x})})$ ,  $\tilde{\mu}_j(\tilde{x}) = \mu_j(x)$ ,  $j = 0, \dots, n$ . The intricate relation between the variable  $x$  and  $\tilde{x}$  is discussed detailedly in Theorem 4.5.

Based on the above all these preparations, let us now give an explicit representations for the meromorphic function  $\phi$  and the stationary HS solutions  $u$  in terms of the Riemann theta function associated with  $\mathcal{K}_n$ . Here we assume the affine part of  $\mathcal{K}_n$  to be nonsingular.

**Theorem 4.5** *Assume that the curve  $\mathcal{K}_n$  is nonsingular, (2.1) holds and  $u$  satisfies the  $n$ th stationary HS equation (2.34) on  $\Omega$ . Moreover, let  $P = (z, y) \in \mathcal{K}_n \setminus \{P_0\}$ , and  $x \in \Omega$ , where  $\Omega \subseteq \mathbb{R}$  is an open interval. In addition, suppose that  $\mathcal{D}_{\hat{\mu}(x)}$ , or equivalently  $\mathcal{D}_{\hat{\mu}(x)}$  is nonspecial for  $x \in \Omega$ . Then,  $\phi$  and  $u$  have the following representations*

$$\begin{aligned} \phi(P, x) &= ia \left( \prod_{m=1}^{2n+1} E_m \right)^{1/2} f_{n+1}^{-1} \frac{\theta(\underline{z}(P, \hat{\mu}(x))) \theta(\underline{z}(P_0, \hat{\mu}(x)))}{\theta(\underline{z}(P_0, \hat{\mu}(x))) \theta(\underline{z}(P, \hat{\mu}(x)))} \\ &\times \exp \left( d_0 - \int_{Q_0}^P \omega_{\hat{\mu}_0(x)P_0}^{(3)} \right), \end{aligned} \quad (4.34)$$

$$\begin{aligned}
u(x) &= -\frac{1}{2} \sum_{m=0}^{2n+1} E_m + \frac{1}{2} \sum_{j=1}^n \lambda_j \\
&\quad - \frac{1}{2} \sum_{j=1}^n U_j \partial_{\omega_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty+}, \underline{\hat{\mu}}(x)) + \underline{\omega})}{\theta(\underline{z}(P_{\infty-}, \underline{\hat{\mu}}(x)) + \underline{\omega})} \right) \Big|_{\underline{\omega}=0}. \quad (4.35)
\end{aligned}$$

Moreover, let  $\mu_j$ ,  $j = 0, \dots, n$  be not vanishing on  $\Omega$  and  $x, x_0 \in \Omega$ . Then, we have the following constraint

$$\begin{aligned}
2a(x - x_0) &= -2a \int_{x_0}^x \frac{dx'}{\prod_{k=0}^n \mu_k(x')} \sum_{j=1}^n \left( \int_{a_j} \tilde{\omega}_{P_{\infty+} P_{\infty-}}^{(3)} \right) c_j(1) \\
&\quad + \ln \left( \frac{\theta(\underline{z}(P_{\infty-}, \underline{\hat{\mu}}(x_0))) \theta(\underline{z}(P_{\infty+}, \underline{\hat{\mu}}(x)))}{\theta(\underline{z}(P_{\infty+}, \underline{\hat{\mu}}(x_0))) \theta(\underline{z}(P_{\infty-}, \underline{\hat{\mu}}(x)))} \right) \quad (4.36)
\end{aligned}$$

and

$$\begin{aligned}
\hat{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x)} \underline{\hat{\mu}}(x)) &= \hat{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x_0)} \underline{\hat{\mu}}(x_0)) - 2a \int_{x_0}^x \frac{dx'}{\Psi_{n+1}(\underline{\mu}(x'))} \underline{c}(1) \\
&= \hat{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x_0)} \underline{\hat{\mu}}(x_0)) - \underline{c}(1)(\tilde{x} - \tilde{x}_0). \quad (4.37)
\end{aligned}$$

**Proof.** First, let us assume

$$\mu_j(x) \neq \mu_{j'}(x), \quad \nu_k(x) \neq \nu_{k'}(x) \quad \text{for } j \neq j', k \neq k' \text{ and } x \in \tilde{\Omega}, \quad (4.38)$$

where  $\tilde{\Omega} \subseteq \Omega$ . From (3.14),  $\mathcal{D}_{P_0 \hat{\nu}} \sim \mathcal{D}_{\hat{\mu}_0 \hat{\mu}}$ , and  $(P_0)^* \notin \{\hat{\nu}_1, \dots, \hat{\nu}_n\}$  by hypothesis, one can use Theorem A.31 [15] to conclude that  $\mathcal{D}_{\hat{\mu}} \in \text{Sym}^n(\mathcal{K}_n)$  is nonspecial. This argument is of course symmetric with respect to  $\underline{\hat{\mu}}$  and  $\underline{\hat{\nu}}$ . Thus,  $\mathcal{D}_{\hat{\mu}}$  is nonspecial if and only if  $\mathcal{D}_{\hat{\nu}}$  is.

Next, we derive the representations of  $\phi$  and  $u$  in terms of the Riemann theta function. A special case of Riemann's vanishing theorem (Theorem A.26 [15]) yields

$$\theta(\Xi_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}})) = 0 \quad \text{if and only if } P \in \{Q_1, \dots, Q_n\}. \quad (4.39)$$

Therefore, the divisor (3.14) of  $\phi(P, x)$  suggests considering expressions of the following type

$$C(x) \frac{\theta(\Xi_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x)}))}{\theta(\Xi_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}))} \exp\left(d_0 - \int_{Q_0}^P \omega_{\hat{\mu}_0(x) P_0}^{(3)}\right), \quad (4.40)$$

where  $C(x)$  is independent of  $P \in \mathcal{K}_n$ . So, together with the asymptotic expansion of  $\phi(P, x)$  near  $P_0$  in (4.2), we are able to obtain (4.34). The



representation (4.35) for  $u$  on  $\tilde{\Omega}$  follows from trace formula (3.46) and the expression (F.88 [15]) for  $\sum_{j=0}^n \mu_j$ .

To prove the constraint (4.36), one can refs Theorem 4.5 in our latest paper [18]. Equations (4.37) is clear from (4.15). Finally, the extension of all results from  $x \in \tilde{\Omega}$  to  $x \in \Omega$  follows by the continuity of  $\underline{\alpha}_{Q_0}$  and the hypothesis of  $\mathcal{D}_{\hat{\mu}(x)}$  being nonspecial for  $x \in \Omega$ .  $\square$

**Remark 4.6** *The stationary HS solutions  $u$  in (4.35) is a quasi-periodic function with respect to the new variable  $\tilde{x}$  in (4.33). The Abel map in (4.37) linearize the divisor  $\mathcal{D}_{\hat{\mu}_0(x)\hat{\mu}(x)}$  on  $\Omega$  with respect to  $\tilde{x}$ .*

**Remark 4.7** *The similar results to (4.36) and (4.37) (i.e. the Abel map also linearize the divisor  $\mathcal{D}_{\hat{\nu}(x)}$  on  $\Omega$  with respect to  $\bar{x}$ ) hold for the divisor  $\mathcal{D}_{\hat{\nu}(x)}$  associated with  $\phi(P, x)$ . The change of variables is*

$$x \mapsto \bar{x} = \int^x dx' \left( \frac{1}{\Psi_n(\underline{\nu}(x'))} \frac{u_{x'x'}}{h_0(x')} \right). \quad (4.41)$$

**Remark 4.8** *Since  $\mathcal{D}_{P_0\hat{\nu}}$  and  $\mathcal{D}_{\hat{\mu}_0\hat{\mu}}$  are linearly equivalent, that is*

$$\underline{A}_{Q_0}(\hat{\mu}_0(x)) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}(x)}) = \underline{A}_{Q_0}(P_0) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\nu}(x)}). \quad (4.42)$$

*Then we infer*

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\nu}(x)}) = \underline{\Delta} + \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}(x)}), \quad \underline{\Delta} = \underline{A}_{P_0}(\hat{\mu}_0(x)). \quad (4.43)$$

*Hence*

$$\underline{z}(P, \hat{\nu}) = \underline{z}(P, \hat{\mu}) + \underline{\Delta}, \quad P \in \mathcal{K}_n. \quad (4.44)$$

*The representations of  $\phi$  and  $u$  in (4.34) and (4.35) can be rewritten in terms of  $\mathcal{D}_{\hat{\nu}(x)}$  respectively.*

**Remark 4.9** *We have emphasized in Remark 4.2 that the Baker-Akhiezer functions  $\psi$  in (3.6) and (3.8) for the HS hierarchy enjoy very difference from standard Baker-Akhiezer functions. Hence, one may not expect the usual theta function representations of  $\psi_j$ ,  $j = 1, 2$ , in terms of ratios of theta functions times a exponential term including  $(x - x_0)$  multiplying a meromorphic differential with a pole at the essential singularity of  $\psi_j$ . However, using the properties of symmetric function and (F.89) [15], we obtain*

$$\begin{aligned} F_{n+1}(z) &= z^{n+1} + \sum_{k=0}^n \Psi_{n+1-k}(\bar{\mu}) z^k \\ &= z^{n+1} + \sum_{k=1}^n \left( \Psi_{n+1-k}(\underline{\Delta}) \right) \end{aligned}$$

$$\begin{aligned}
& - \sum_{j=1}^n c_j(k) \partial_{\omega_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}) + \underline{\omega})}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}) + \underline{\omega})} \right) \Big|_{\underline{\omega}=0} \Big) z^k \\
& = z \prod_{j=1}^n (z - \lambda_j) \\
& - \sum_{j=1}^n \sum_{k=1}^n c_j(k) \partial_{\omega_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}) + \underline{\omega})}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}) + \underline{\omega})} \right) \Big|_{\underline{\omega}=0} z^k, \quad (4.45)
\end{aligned}$$

and by inserting (4.45) into (3.26), we obtain the theta function representation of  $\psi_1$ . Then, the corresponding theta functions representation of  $\psi_2$  follows by (3.8) and (4.34).

At the end of this section, we turn to the initial value problem in the stationary case. We show that the solvability of the Dubrovin equations (3.38) on  $\Omega_\mu \subseteq \mathbb{R}$  in fact implies the stationary HS equation (2.34) on  $\Omega_\mu$ , which amounts to solving the algebro-geometric initial value problem in the stationary case.

**Theorem 4.10** *Assume that (2.1) holds and  $\{\hat{\mu}_j\}_{j=0,\dots,n}$  satisfies the stationary Dubrovin equations (3.38) on  $\Omega_\mu$  and remain distinct and nonzero for  $x \in \Omega_\mu$ , where  $\Omega_\mu \subseteq \mathbb{R}$  is an open interval. Then,  $u$  defined by*

$$u = -\frac{1}{2} \sum_{m=0}^{2n+1} E_m + \frac{1}{2} \sum_{j=0}^n \mu_j, \quad (4.46)$$

satisfies the  $n$ th stationary HS equation (2.34), that is

$$\text{s-HS}_n(u) = 0, \quad \text{on } \Omega_\mu. \quad (4.47)$$

**Proof.** Given the solutions  $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in C^\infty(\Omega_\mu, \mathcal{K}_n)$ ,  $j = 0, \dots, n$  of (3.38), let us introduce

$$F_{n+1}(z) = \prod_{j=0}^n (z - \mu_j) \quad \text{on } \mathbb{C} \times \Omega_\mu, \quad (4.48)$$

with  $u$  defined by (4.46) up to multiplicative constant. Given  $F_{n+1}$  and  $u$ , let us denote the polynomial  $G_n$  by

$$G_n(z) = \frac{1}{2} F_{n+1,x}(z), \quad \text{on } \mathbb{C} \times \Omega_\mu, \quad (4.49)$$

and from (4.48), one can see that the degree of  $G_n$  is  $n$  with respect to  $z$ . Taking account into (4.48), the Dubrovin equations (3.38) imply

$$y(\hat{\mu}_j) = \frac{1}{2}\mu_j\mu_{j,x} \prod_{\substack{k=0 \\ k \neq j}}^n (\mu_j - \mu_k) = -\frac{1}{2}\mu_j F_{n+1,x}(\mu_j) = -\mu_j G_n(\mu_j). \quad (4.50)$$

Hence

$$R_{2n+2}(\mu_j)^2 - \mu_j^2 G_n(\mu_j)^2 = y(\hat{\mu}_j)^2 - \mu_j^2 G_n(\mu_j)^2 = 0, \quad j = 0, \dots, n. \quad (4.51)$$

Next, let us define a polynomial  $H_n$  on  $\mathbb{C} \times \Omega_\mu$  such that

$$R_{2n+2}(z) - z^2 G_n(z)^2 = z F_{n+1}(z) H_n(z) \quad (4.52)$$

holds. Such a polynomial  $H_n$  exists since the left-hand side of (4.52) vanishes at  $z = \mu_j$ ,  $j = 0, \dots, n$  by (4.51). We need to determine the degree of  $H_n$ . By (4.49), we compute

$$R_{2n+2}(z) - z^2 G_n(z)^2 \underset{|z| \rightarrow \infty}{=} h_0 z^{2n+2} + O(z^{2n+1}), \quad (4.53)$$

with  $O(z^{2n+1})$  depending on  $x$  by inspection. Therefore, combining (4.48), (4.49), (4.52) and (4.53), we conclude that  $H_n$  has degree  $n$  with respect to  $z$ , with the coefficient  $h_0$  of powers  $z^n$ . Hence, we may write  $H_n$  as

$$H_n(z) = h_0 \prod_{l=1}^n (z - \nu_l), \quad \text{on } \mathbb{C} \times \Omega_\mu. \quad (4.54)$$

Next, let us consider the polynomial  $P_n$  by

$$P_n(z) = H_n(z) + u_{xx} F_{n+1}(z) + z G_{n,x}(z). \quad (4.55)$$

Using (4.48), (4.49) and (4.54) we obtain that  $P_n$  is a polynomial of degree at most  $n$ . Differentiating on both sides of (4.52) with respect to  $x$  yields

$$2z^2 G_n(z) G_{n,x}(z) + z F_{n+1,x}(z) H_n(z) + z F_{n+1}(z) H_{n,x}(z) = 0 \quad \text{on } \mathbb{C} \times \Omega_\mu. \quad (4.56)$$

Multiplying (4.55) by  $G_n$  and using (4.56), we have

$$\begin{aligned} G_n(z) P_n(z) &= F_{n+1}(z) (u_{xx} G_n(z) - \frac{1}{2} H_{n,x}(z)) \\ &\quad + (G_n(z) - \frac{1}{2} F_{n+1,x}(z)) H_n(z), \end{aligned} \quad (4.57)$$

and hence

$$G_n(\mu_j)P_n(\mu_j) = 0, \quad j = 1, \dots, n, \quad (4.58)$$

on  $\Omega_\mu$  by using (4.49).

Next, let  $x \in \tilde{\Omega}_\mu \subseteq \Omega_\mu$ , where  $\tilde{\Omega}_\mu$  is given by

$$\begin{aligned} \tilde{\Omega}_\mu &= \{x \in \Omega_\mu \mid G(\mu_j(x), x) = -\frac{y(\hat{\mu}_j(x))}{\mu_j(x)} \neq 0, j = 0, \dots, n\} \\ &= \{x \in \Omega_\mu \mid \mu_j(x) \notin \{E_m\}_{m=0, \dots, 2n+1}, j = 0, \dots, n\}, \end{aligned} \quad (4.59)$$

Thus, we have

$$P_n(\mu_j(x), x) = 0, \quad j = 0, \dots, n, \quad x \in \tilde{\Omega}_\mu. \quad (4.60)$$

Since  $P_n$  is a polynomial of degree at most  $n$ , (4.60) implies

$$P_n = 0 \quad \text{on } \mathbb{C} \times \tilde{\Omega}_\mu, \quad (4.61)$$

So, (2.17) holds, that is,

$$zG_{n,x}(z) = -H_n(z) - u_{xx}F_{n+1}(z) \quad \text{on } \mathbb{C} \times \tilde{\Omega}_\mu. \quad (4.62)$$

Inserting (4.62) and (4.49) into (4.56) yields

$$zF_{n+1}(z)(-2u_{xx}G_n(z) + H_{n,x}(z)) = 0, \quad (4.63)$$

namely

$$H_{n,x}(z) = 2u_{xx}G_n(z), \quad \text{on } \mathbb{C} \times \tilde{\Omega}_\mu. \quad (4.64)$$

Thus, we obtain the fundamental equations (2.15)-(2.17), and (2.19) on  $\mathbb{C} \times \tilde{\Omega}_\mu$ .

In order to extend these results to all  $x \in \Omega_\mu$ , let us consider the case where  $\hat{\mu}_j$  admits one of the branch points  $(E_{m_0}, 0)$ . Hence, we suppose

$$\mu_{j_1}(x) \rightarrow E_{m_0} \quad \text{as } x \rightarrow x_0 \in \Omega_\mu, \quad (4.65)$$

for some  $j_1 \in \{0, \dots, n\}$ ,  $m_0 \in \{1, \dots, 2n+1\}$ . Introducing

$$\begin{aligned} \zeta_{j_1}(x) &= \sigma(\mu_{j_1}(x) - E_{m_0})^{1/2}, \quad \sigma = \pm 1, \\ \mu_{j_1}(x) &= E_{m_0} + \zeta_{j_1}(x)^2 \end{aligned} \quad (4.66)$$

for some  $x$  in an open interval centered near  $x_0$ , then the Dubrovin equation (3.38) for  $\mu_{j_1}$  becomes

$$\begin{aligned} \zeta_{j_1, x}(x) &= c(\sigma) \frac{a}{E_{m_0}} \left( \prod_{\substack{m=0 \\ m \neq m_0}}^{2n+1} (E_{m_0} - E_m) \right)^{1/2} \\ &\times \prod_{\substack{k=0 \\ k \neq j_1}}^n (E_{m_0} - \mu_k(x))^{-1} (1 + O(\zeta_{j_1}(x)^2)) \end{aligned} \quad (4.67)$$

for some  $|c(\sigma)| = 1$ . Hence (4.61)-(4.64) extend to  $\Omega_\mu$  by continuity. Consequently, we obtain relations (2.15)-(2.17) on  $\mathbb{C} \times \Omega_\mu$ , and can proceed as in Section 2 to see that  $u$  satisfies the stationary HS hierarchy (4.47).  $\square$

**Remark 4.11** *The result in Theorem 4.10 is derived in terms of  $u$  and  $\{\mu_j\}_{j=0, \dots, n}$ , but one can prove the analogous result in terms of  $u$  and  $\{\nu_l\}_{l=1, \dots, n}$ .*

**Remark 4.12** *Theorem 4.10 reveals that given  $\mathcal{K}_n$  and the initial condition  $(\hat{\mu}_0(x_0), \hat{\mu}_1(x_0), \dots, \hat{\mu}_n(x_0))$ , or equivalently, the auxiliary divisor  $\mathcal{D}_{\hat{\mu}_0(x_0)\hat{\mu}(x_0)}$  at  $x = x_0$ ,  $u$  is uniquely determined in an open neighborhood  $\Omega$  of  $x_0$  by (4.46) and satisfies the  $n$ th stationary HS equation (2.34). Conversely, given  $\mathcal{K}_n$  and  $u$  in an open neighborhood  $\Omega$  of  $x_0$ , we can construct the corresponding polynomial  $F_{n+1}(z, x)$ ,  $G_n(z, x)$  and  $H_n(z, x)$  for  $x \in \Omega$ , and then obtain the auxiliary divisor  $\mathcal{D}_{\hat{\mu}_0(x)\hat{\mu}(x)}$  for  $x \in \Omega$  from the zeros of  $F_{n+1}(z, x)$  and (3.10). In that sense, once the curve  $\mathcal{K}_n$  is fixed, elements of the isospectral class of the HS potentials  $u$  can be characterized by nonspecial auxiliary divisor  $\mathcal{D}_{\hat{\mu}_0(x)\hat{\mu}(x)}$ .*

## 5 The time-dependent HS formalism

In this section, let us go back to the recursive approach detailed in Section 2 and extend the the algebro-geometric analysis of Section 3 to the time-dependent HS hierarchy.

Throughout this section we assume (2.2) to hold.

The time-dependent algebro-geometric initial value problem of the HS hierarchy is to solve the time-dependent  $r$ th HS flow with a stationary solution of the  $n$ th equation as initial data in the hierarchy. More precisely, given  $n \in \mathbb{N}_0$ , based on the solution  $u^{(0)}$  of the  $n$ th stationary HS equation  $s\text{-HS}_n(u^{(0)}) = 0$  associated with  $\mathcal{K}_n$  and a set of integration constants  $\{c_l\}_{l=1, \dots, n} \subset \mathbb{C}$ , we want to build up a solution  $u$  of the  $r$ th HS flow  $\text{HS}_r(u) = 0$  such that  $u(t_{0,r}) = u^{(0)}$  for some  $t_{0,r} \in \mathbb{R}$ ,  $r \in \mathbb{N}_0$ .

We employ the notations  $\tilde{V}_r, \tilde{F}_{r+1}, \tilde{G}_r, \tilde{H}_r, \tilde{f}_s, \tilde{g}_s, \tilde{h}_s$  to stand for the time-dependent quantities, which are obtained in  $V_n, F_{n+1}, G_n, H_n, f_l, g_l, h_l$  by replacing  $\{c_l\}_{l=1,\dots,n}$  with  $\{\tilde{c}_s\}_{s=1,\dots,r}$ , where the integration constants  $\{c_l\}_{l=1,\dots,n} \subset \mathbb{C}$  in the stationary HS hierarchy and  $\{\tilde{c}_s\}_{s=1,\dots,r} \subset \mathbb{C}$  in the time-dependent HS hierarchy are independent of each other. In addition, we mark the individual  $r$ th HS flow by a separate time variable  $t_r \in \mathbb{R}$ .

Let us now provide the time-dependent algebro-geometric initial value problem as follows

$$\text{HS}_r(u) = -u_{xxt_r} + u_{xxx}\tilde{f}_{r+1}(u) + 2u_{xx}\tilde{f}_{r+1,x}(u) = 0, \quad (5.1)$$

$$u|_{t_r=t_{0,r}} = u^{(0)},$$

$$\text{s-HS}_n(u^{(0)}) = u_{xxx}f_{n+1}(u^{(0)}) + 2u_{xx}f_{n+1,x}(u^{(0)}) = 0, \quad (5.2)$$

where  $t_{0,r} \in \mathbb{R}$ ,  $n, r \in \mathbb{N}_0$ ,  $u = u(x, t_r)$  satisfies the condition (2.2), and the curve  $\mathcal{K}_n$  is associated with the initial data  $u^{(0)}$  in (5.2). Noticing that the HS flows are isospectral, we are going a further step and assume that (5.2) holds not only at  $t_r = t_{0,r}$ , but also at all  $t_r \in \mathbb{R}$ .

Let us now start from the zero-curvature equations (2.41)

$$U_{t_r} - \tilde{V}_{r,x} + [U, \tilde{V}_r] = 0, \quad (5.3)$$

$$-V_{n,x} + [U, V_n] = 0, \quad (5.4)$$

where

$$\begin{aligned} U(z) &= \begin{pmatrix} 0 & 1 \\ -z^{-1}u_{xx} & 0 \end{pmatrix} \\ V_n(z) &= \begin{pmatrix} -G_n(z) & F_{n+1}(z) \\ z^{-1}H_n(z) & G_n(z) \end{pmatrix} \\ \tilde{V}_r(z) &= \begin{pmatrix} -\tilde{G}_r(z) & \tilde{F}_{r+1}(z) \\ z^{-1}\tilde{H}_r(z) & \tilde{G}_r(z) \end{pmatrix} \end{aligned} \quad (5.5)$$

and

$$F_{n+1}(z) = \sum_{l=0}^{n+1} f_l z^{n+1-l} = \prod_{j=0}^n (z - \mu_j), \quad (5.6)$$

$$G_n(z) = \sum_{l=0}^n g_l z^{n-l}, \quad (5.7)$$

$$H_n(z) = \sum_{l=0}^n h_l z^{n-l} = h_0 \prod_{l=1}^n (z - \nu_l), \quad (5.8)$$

$$\tilde{F}_{r+1}(z) = \sum_{s=0}^{r+1} \tilde{f}_s z^{r+1-s}, \quad (5.9)$$

$$\tilde{G}_r(z) = \sum_{s=0}^r \tilde{g}_s z^{r-s}, \quad (5.10)$$

$$\tilde{H}_r(z) = \sum_{s=0}^r \tilde{h}_s z^{r-s}, \quad (5.11)$$

for fixed  $n, r \in \mathbb{N}_0$ . Here  $\{f_l\}_{l=0,\dots,n+1}$ ,  $\{g_l\}_{l=0,\dots,n}$ ,  $\{h_l\}_{l=0,\dots,n}$ , and  $\{\tilde{f}_s\}_{s=0,\dots,r+1}$ ,  $\{\tilde{g}_s\}_{s=0,\dots,r}$ ,  $\{\tilde{h}_s\}_{s=0,\dots,r}$ , satisfy the relations in (2.3).

Moreover, it is more convenient for us to rewrite the zero-curvature equations (5.3) and (5.4) as the following forms,

$$-u_{xxt_r} - \tilde{H}_{r,x} + 2u_{xx}\tilde{G}_r = 0, \quad (5.12)$$

$$\tilde{F}_{r+1,x} = 2\tilde{G}_r, \quad (5.13)$$

$$z\tilde{G}_{r,x} = -\tilde{H}_r - u_{xx}\tilde{F}_{r+1} \quad (5.14)$$

and

$$F_{n+1,x} = 2G_n, \quad (5.15)$$

$$H_{n,x} = 2u_{xx}G_n, \quad (5.16)$$

$$zG_{n,x} = -H_n - u_{xx}F_{n+1}. \quad (5.17)$$

From (5.15)-(5.17), we may compute

$$\frac{d}{dx} \det(V_n(z)) = -\frac{1}{z^2} \frac{d}{dx} \left( z^2 G_n(z)^2 + z F_{n+1}(z) H_n(z) \right) = 0, \quad (5.18)$$

and meanwhile Lemma 5.2 gives

$$\frac{d}{dt_r} \det(V_n(z)) = -\frac{1}{z^2} \frac{d}{dt_r} \left( z^2 G_n(z)^2 + z F_{n+1}(z) H_n(z) \right) = 0, \quad (5.19)$$

Hence,  $z^2 G_n(z)^2 + z F_{n+1}(z) H_n(z)$  is independent of variables both  $x$  and  $t_r$ , which implies

$$z^2 G_n(z)^2 + z F_{n+1}(z) H_n(z) = R_{2n+2}(z). \quad (5.20)$$

This reveals that the fundamental identity (2.19) still holds in the time-dependent context. Consequently the hyperelliptic curve  $\mathcal{K}_n$  is still available by (2.27).

Next, let us introduce the time-dependent Baker-Akhiezer function  $\psi(P, x, x_0, t_r, t_{0,r})$  on  $\mathcal{K}_n \setminus \{P_{\infty\pm}, P_0\}$  by

$$\begin{aligned}\psi(P, x, x_0, t_r, t_{0,r}) &= \begin{pmatrix} \psi_1(P, x, x_0, t_r, t_{0,r}) \\ \psi_2(P, x, x_0, t_r, t_{0,r}) \end{pmatrix}, \\ \psi_x(P, x, x_0, t_r, t_{0,r}) &= U(u(x, t_r), z(P))\psi(P, x, x_0, t_r, t_{0,r}), \\ \psi_{t_r}(P, x, x_0, t_r, t_{0,r}) &= \tilde{V}_r(u(x, t_r), z(P))\psi(P, x, x_0, t_r, t_{0,r}), \\ zV_n(u(x, t_r), z(P))\psi(P, x, x_0, t_r, t_{0,r}) &= y(P)\psi(P, x, x_0, t_r, t_{0,r}), \\ \psi_1(P, x_0, x_0, t_{0,r}, t_{0,r}) &= 1; \\ P = (z, y) &\in \mathcal{K}_n \setminus \{P_{\infty\pm}, P_0\}, \quad (x, t_r) \in \mathbb{R}^2,\end{aligned}\tag{5.21}$$

where

$$\begin{aligned}\psi_1(P, x, x_0, t_r, t_{0,r}) &= \exp\left(\int_{t_{0,r}}^{t_r} ds(z^{-1}\tilde{F}_{r+1}(z, x_0, s)\phi(P, x_0, s) \right. \\ &\quad \left. - \tilde{G}_r(z, x_0, s)) + z^{-1} \int_{x_0}^x dx' \phi(P, x', t_r)\right), \\ P = (z, y) &\in \mathcal{K}_n \setminus \{P_{\infty\pm}, P_0\}.\end{aligned}\tag{5.22}$$

Closely related to  $\psi(P, x, x_0, t_r, t_{0,r})$  is the following meromorphic function  $\phi(P, x, t_r)$  on  $\mathcal{K}_n$  defined by

$$\phi(P, x, t_r) = z \frac{\psi_{1,x}(P, x, x_0, t_r, t_{0,r})}{\psi_1(P, x, x_0, t_r, t_{0,r})}, \quad P \in \mathcal{K}_n \setminus \{P_{\infty\pm}, P_0\}, \quad (x, t_r) \in \mathbb{R}^2.\tag{5.23}$$

which implies by (5.21) that

$$\begin{aligned}\phi(P, x, t_r) &= \frac{y + zG_n(z, x, t_r)}{F_{n+1}(z, x, t_r)} \\ &= \frac{zH_n(z, x, t_r)}{y - zG_n(z, x, t_r)},\end{aligned}\tag{5.24}$$

and

$$\psi_2(P, x, x_0, t_r, t_{0,r}) = \psi_1(P, x, x_0, t_r, t_{0,r})\phi(P, x, t_r)/z.\tag{5.25}$$

In analogy to equations (3.10) and (3.11), we define

$$\begin{aligned}\hat{\mu}_j(x, t_r) &= (\mu_j(x, t_r), -\mu_j(x, t_r)G_n(\mu_j(x, t_r), x, t_r)) \in \mathcal{K}_n, \\ j &= 0, \dots, n, \quad (x, t_r) \in \mathbb{R}^2,\end{aligned}\tag{5.26}$$



$$\begin{aligned}\hat{\nu}_l(x, t_r) &= (\nu_l(x, t_r), \nu_l(x, t_r)G_n(\nu_l(x, t_r), x, t_r)) \in \mathcal{K}_n, \\ l &= 1, \dots, n, (x, t_r) \in \mathbb{R}^2.\end{aligned}\quad (5.27)$$

The regular properties of  $F_{n+1}$ ,  $H_n$ ,  $\mu_j$  and  $\nu_l$  are analogous to those in Section 3 due to assumptions (2.2).

From (5.24), the the divisor  $(\phi(P, x, t_r))$  of  $\phi(P, x, t_r)$  reads

$$(\phi(P, x, t_r)) = \mathcal{D}_{P_0\hat{\mu}(x, t_r)}(P) - \mathcal{D}_{\hat{\mu}_0(x, t_r)\hat{\mu}(x, t_r)}(P) \quad (5.28)$$

where

$$\hat{\mu} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}, \quad \hat{\nu} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \text{Sym}^n(\mathcal{K}_n). \quad (5.29)$$

That means  $P_0, \hat{\nu}_1(x, t_r), \dots, \hat{\nu}_n(x, t_r)$  are the  $n+1$  zeros of  $\phi(P, x, t_r)$  and  $\hat{\mu}_0(x, t_r), \hat{\mu}_1(x, t_r), \dots, \hat{\mu}_n(x, t_r)$  its  $n+1$  poles.

Further properties of  $\phi(P, x, t_r)$  are summarized as follows.

**Lemma 5.1** *Assume that (2.2), (5.3) and (5.4) hold. Let  $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty\pm}, P_0\}$ ,  $(x, t_r) \in \mathbb{R}^2$ . Then*

$$\phi_x(P) + z^{-1}\phi(P)^2 = -u_{xx}, \quad (5.30)$$

$$\begin{aligned}\phi_{t_r}(P) &= (-z\tilde{G}_r(z) + \tilde{F}_{r+1}(z)\phi(P))_x \\ &= \tilde{H}_r(z) + u_{xx}\tilde{F}_{r+1}(z) + (\tilde{F}_{r+1}(z)\phi(P))_x,\end{aligned}\quad (5.31)$$

$$\phi_{t_r}(P) = \tilde{H}_r(z) + 2\tilde{G}_r(z)\phi(P) - z^{-1}\tilde{F}_{r+1}(z)\phi(P)^2, \quad (5.32)$$

$$\phi(P)\phi(P^*) = -\frac{zH_n(z)}{F_{n+1}(z)}, \quad (5.33)$$

$$\phi(P) + \phi(P^*) = 2\frac{zG_n(z)}{F_{n+1}(z)}, \quad (5.34)$$

$$\phi(P) - \phi(P^*) = \frac{2y}{F_{n+1}(z)}. \quad (5.35)$$

**Proof.** We just need to prove (5.31) and (5.32). Equations (5.30) and (5.33)-(5.35) can be proved as in Lemma 3.1. By using (5.21) and (5.23), we obtain

$$\begin{aligned}\phi_{t_r} &= z(\ln\psi_1)_{xt_r} = z(\ln\psi_1)_{t_rx} = z\left(\frac{\psi_{1,t_r}}{\psi_1}\right)_x \\ &= z\left(\frac{-\tilde{G}_r\psi_1 + \tilde{F}_{r+1}\psi_2}{\psi_1}\right)_x \\ &= (-z\tilde{G}_r + \tilde{F}_{r+1}\phi)_x,\end{aligned}\quad (5.36)$$

which is the first line of (5.31). Inserting (5.14) into (5.36) yields the second line of (5.31). Then by the definition of  $\phi$  (5.23), one may have

$$\begin{aligned}
\phi_{t_r} &= z \left( \frac{\psi_2}{\psi_1} \right)_{t_r} = z \left( \frac{\psi_{2,t_r}}{\psi_1} - \frac{\psi_2 \psi_{1,t_r}}{\psi_1^2} \right) \\
&= z \left( \frac{z^{-1} \tilde{H}_r \psi_1 + \tilde{G}_r \psi_2}{\psi_1} - z^{-1} \phi \frac{-\tilde{G}_r \psi_1 + \tilde{F}_{r+1} \psi_2}{\psi_1} \right) \\
&= \tilde{H}_r + 2\tilde{G}_r \phi - z^{-1} \tilde{F}_{r+1} \phi^2,
\end{aligned} \tag{5.37}$$

which is (5.32). Alternatively, one can insert (5.12)-(5.14) into (5.31) to obtain (5.32).  $\square$

Next we study the time evolution of  $F_{n+1}$ ,  $G_n$  and  $H_n$  by using zero-curvature equations (5.12)-(5.14) and (5.15)-(5.17).

**Lemma 5.2** *Assume that (2.2), (5.3) and (5.4) hold. Then*

$$F_{n+1,t_r} = 2(G_n \tilde{F}_{r+1} - \tilde{G}_r F_{n+1}), \tag{5.38}$$

$$zG_{n,t_r} = \tilde{H}_r F_{n+1} - H_n \tilde{F}_{r+1}, \tag{5.39}$$

$$H_{n,t_r} = 2(H_n \tilde{G}_r - G_n \tilde{H}_r). \tag{5.40}$$

Equations (5.38) – (5.40) imply

$$-V_{n,t_r} + [\tilde{V}_r, V_n] = 0. \tag{5.41}$$

**Proof.** Differentiating both sides of (5.35) with respect to  $t_r$  leads to

$$(\phi(P) - \phi(P^*))_{t_r} = -2y F_{n+1,t_r} F_{n+1}^{-2}. \tag{5.42}$$

On the other hand, by (5.32), (5.34) and (5.35), the left-hand side of (5.42) equals to

$$\begin{aligned}
\phi(P)_{t_r} - \phi(P^*)_{t_r} &= 2\tilde{G}_r(\phi(P) - \phi(P^*)) - z^{-1} \tilde{F}_{r+1}(\phi(P)^2 - \phi(P^*)^2) \\
&= 4y(\tilde{G}_r F_{n+1} - \tilde{F}_{r+1} G_n) F_{n+1}^{-2}.
\end{aligned} \tag{5.43}$$

Combining (5.42) with (5.43) yields (5.38). Similarly, Differentiating both sides of (5.34) with respect to  $t_r$  gives

$$(\phi(P) + \phi(P^*))_{t_r} = 2z(G_{n,t_r} F_{n+1} - G_n F_{n+1,t_r}) F_{n+1}^{-2}, \tag{5.44}$$

Meanwhile, by (5.32), (5.33) and (5.34), the left-hand side of (5.44) equals to

$$\begin{aligned}\phi(P)_{t_r} + \phi(P^*)_{t_r} &= 2\tilde{G}_r(\phi(P) + \phi(P^*)) - z^{-1}\tilde{F}_{r+1}(\phi(P)^2 + \phi(P^*)^2) + 2\tilde{H}_r \\ &= -2zG_nF_{n+1}^{-2}F_{n+1,t_r} + 2F_{n+1}^{-1}(\tilde{H}_rF_{n+1} - \tilde{F}_{r+1}H_n). \end{aligned} \quad (5.45)$$

Thus, (5.39) clearly follows by (5.44) and (5.45). Hence, insertion of (5.38) and (5.39) into the differentiation of  $z^2G_n^2 + zF_{n+1}H_n = R_{2n+2}(z)$  can derive (5.40). Finally, a direct calculation shows that (5.38)-(5.40) are equivalent to (5.41).  $\square$

Further properties of  $\psi$  are summarized as follows.

**Lemma 5.3** *Assume that (2.2), (5.3) and (5.4) hold. Let  $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty\pm}, P_0\}$ ,  $(x, x_0, t_r, t_{0,r}) \in \mathbb{R}^4$ . Then, we have*

$$\begin{aligned}\psi_1(P, x, x_0, t_r, t_{0,r}) &= \left( \frac{F_{n+1}(z, x, t_r)}{F_{n+1}(z, x_0, t_{0,r})} \right)^{1/2} \\ &\times \exp\left( \frac{y}{z} \int_{t_{0,r}}^{t_r} ds \tilde{F}_{r+1}(z, x_0, s) F_{n+1}(z, x_0, s)^{-1} \right. \\ &\quad \left. + \frac{y}{z} \int_{x_0}^x dx' F_{n+1}(z, x', t_r)^{-1} \right), \end{aligned} \quad (5.46)$$

$$\psi_1(P, x, x_0, t_r, t_{0,r}) \psi_1(P^*, x, x_0, t_r, t_{0,r}) = \frac{F_{n+1}(z, x, t_r)}{F_{n+1}(z, x_0, t_{0,r})}, \quad (5.47)$$

$$\psi_2(P, x, x_0, t_r, t_{0,r}) \psi_2(P^*, x, x_0, t_r, t_{0,r}) = -\frac{H_n(z, x, t_r)}{zF_{n+1}(z, x_0, t_{0,r})}, \quad (5.48)$$

$$\psi_1(P, x, x_0, t_r, t_{0,r}) \psi_2(P^*, x, x_0, t_r, t_{0,r}) \quad (5.49)$$

$$+ \psi_1(P^*, x, x_0, t_r, t_{0,r}) \psi_2(P, x, x_0, t_r, t_{0,r}) = 2 \frac{G_n(z, x, t_r)}{F_{n+1}(z, x_0, t_{0,r})},$$

$$\psi_1(P, x, x_0, t_r, t_{0,r}) \psi_2(P^*, x, x_0, t_r, t_{0,r}) \quad (5.50)$$

$$- \psi_1(P^*, x, x_0, t_r, t_{0,r}) \psi_2(P, x, x_0, t_r, t_{0,r}) = -\frac{2y}{zF_{n+1}(z, x_0, t_{0,r})}.$$

**Proof.** In order to prove (5.46), let us first consider the part of time variable in the definition (5.22), that is

$$\exp\left( \int_{t_{0,r}}^{t_r} ds (z^{-1}\tilde{F}_{r+1}(z, x_0, s)\phi(P, x_0, s) - \tilde{G}_r(z, x_0, s)) \right). \quad (5.51)$$

The integrand in the above integral equals to

$$\begin{aligned}
& z^{-1} \tilde{F}_{r+1}(z, x_0, s) \phi(P, x_0, s) - \tilde{G}_r(z, x_0, s) \\
&= z^{-1} \tilde{F}_{r+1}(z, x_0, s) \frac{y + z G_n(z, x_0, s)}{F_{n+1}(z, x_0, s)} - \tilde{G}_r(z, x_0, s) \\
&= \frac{y}{z} \tilde{F}_{r+1}(z, x_0, s) F_{n+1}(z, x_0, s)^{-1} + (\tilde{F}_{r+1}(z, x_0, s) G_n(z, x_0, s) \\
&\quad - \tilde{G}_r(z, x_0, s) F_{n+1}(z, x_0, s)) F_{n+1}(z, x_0, s)^{-1} \\
&= \frac{y}{z} \tilde{F}_{r+1}(z, x_0, s) F_{n+1}(z, x_0, s)^{-1} + \frac{1}{2} \frac{F_{n+1,s}(z, x_0, s)}{F_{n+1}(z, x_0, s)}, \tag{5.52}
\end{aligned}$$

where we used (5.24) and (5.38). By (5.52), (5.51) reads

$$\left( \frac{F_{n+1}(z, x_0, t_r)}{F_{n+1}(z, x_0, t_{0,r})} \right)^{1/2} \exp \left( \frac{y}{z} \int_{t_{0,r}}^{t_r} ds \tilde{F}_{r+1}(z, x_0, s) F_{n+1}(z, x_0, s)^{-1} \right). \tag{5.53}$$

On the other hand, the part of space variable in (5.22) can be written as

$$\left( \frac{F_{n+1}(z, x, t_r)}{F_{n+1}(z, x_0, t_r)} \right)^{1/2} \exp \left( \frac{y}{z} \int_{x_0}^x dx' F_{n+1}(z, x', t_r)^{-1} \right), \tag{5.54}$$

which can be proved using the similar procedure to Lemma 3.2. Combining (5.53) and (5.54) yields (5.46). Evaluating (5.46) at the points  $P$  and  $P^*$ , and multiplying the resulting expressions, with noticing

$$y(P) + y(P^*) = 0, \tag{5.55}$$

leads to (5.47). The remaining statements (5.48)-(5.50) are direct consequence of (5.25), (5.33)-(5.35) and (5.47).  $\square$

In analogy to Lemma 3.4, the dynamics of the zeros  $\{\mu_j(x, t_r)\}_{j=0,\dots,n}$  and  $\{\nu_l(x, t_r)\}_{l=1,\dots,n}$  of  $F_{n+1}(z, x, t_r)$  and  $H_n(z, x, t_r)$  with respect to  $x$  and  $t_r$  are described in terms of Dubrovin-type equations (see the following Lemma). We assume that the affine part of  $\mathcal{K}_n$  to be nonsingular, which implies (3.37) holds in present context.

**Lemma 5.4** *Assume that (2.2), (5.3) and (5.4) hold.*

(i) *Suppose that the zeros  $\{\mu_j(x, t_r)\}_{j=0,\dots,n}$  of  $F_{n+1}(z, x, t_r)$  remain distinct for  $(x, t_r) \in \Omega_\mu$ , where  $\Omega_\mu \subseteq \mathbb{R}^2$  is open and connected. Then,  $\{\mu_j(x, t_r)\}_{j=0,\dots,n}$  satisfy the system of differential equations,*

$$\mu_{j,x} = 2 \frac{y(\hat{\mu}_j)}{\mu_j} \prod_{\substack{k=0 \\ k \neq j}}^n (\mu_j - \mu_k)^{-1}, \quad j = 0, \dots, n, \tag{5.56}$$

$$\mu_{j,t_r} = \frac{2\tilde{F}_{r+1}(\mu_j)y(\hat{\mu}_j)}{\mu_j} \prod_{\substack{k=0 \\ k \neq j}}^n (\mu_j - \mu_k)^{-1}, \quad j = 0, \dots, n, \quad (5.57)$$

with initial conditions

$$\{\hat{\mu}_j(x_0, t_{0,r})\}_{j=0,\dots,n} \in \mathcal{K}_n, \quad (5.58)$$

for some fixed  $(x_0, t_{0,r}) \in \Omega_\mu$ . The initial value problem (5.57), (5.58) has a unique solution satisfying

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 0, \dots, n. \quad (5.59)$$

(ii) Suppose that the zeros  $\{\nu_l(x, t_r)\}_{l=1,\dots,n}$  of  $H_n(z, x, t_r)$  remain distinct for  $(x, t_r) \in \Omega_\nu$ , where  $\Omega_\nu \subseteq \mathbb{R}^2$  is open and connected. Then,  $\{\nu_l(x, t_r)\}_{l=1,\dots,n}$  satisfy the system of differential equations,

$$\nu_{l,x} = -2 \frac{u_{xx}}{h_0} \frac{y(\hat{\nu}_l)}{\nu_l} \prod_{\substack{k=1 \\ k \neq l}}^n (\nu_l - \nu_k)^{-1}, \quad l = 1, \dots, n, \quad (5.60)$$

$$\nu_{l,t_r} = \frac{2\tilde{H}_r(\nu_l)y(\hat{\nu}_l)}{h_0 \nu_l} \prod_{\substack{k=1 \\ k \neq l}}^n (\nu_l - \nu_k)^{-1}, \quad l = 1, \dots, n, \quad (5.61)$$

with initial conditions

$$\{\hat{\nu}_l(x_0, t_{0,r})\}_{l=1,\dots,n} \in \mathcal{K}_n, \quad (5.62)$$

for some fixed  $(x_0, t_{0,r}) \in \Omega_\nu$ . The initial value problem (5.61), (5.62) has a unique solution satisfying

$$\hat{\nu}_l \in C^\infty(\Omega_\nu, \mathcal{K}_n), \quad l = 1, \dots, n. \quad (5.63)$$

**Proof.** It suffices to focus on (5.56), (5.57) and (5.59), since the proof procedure for (5.60), (5.61) and (5.63) is similar.

The proof of (5.56) has been given in Lemma 3.4. We just derive (5.57). Differentiating on both sides of (5.6) with respect to  $t_r$  yields

$$F_{n+1,t_r}(\mu_j) = -\mu_{j,t_r} \prod_{\substack{k=0 \\ k \neq j}}^n (\mu_j - \mu_k). \quad (5.64)$$

On the other hand, inserting  $z = \mu_j$  into (5.38) and considering (5.26), we arrive at

$$F_{n+1,t_r}(\mu_j) = 2G_n(\mu_j)\tilde{F}_{r+1}(\mu_j) = 2\frac{y(\hat{\mu}_j)}{-\mu_j}\tilde{F}_{r+1}(\mu_j). \quad (5.65)$$

Combining (5.64) with (5.65) leads to (5.57). The proof of smoothness assertion (5.59) is analogous to the mCH case in our latest paper [18].  $\square$

Let us now present the  $t_r$ -dependent trace formulas of HS hierarchy, which are used to construct the algebro-geometric solutions  $u$  in section 6. For simplicity, we just take the simplest case.

**Lemma 5.5** *Assume that (2.2), (5.3) and (5.4) hold. Then, we have*

$$u = \frac{1}{2} \sum_{j=0}^n \mu_j - \frac{1}{2} \sum_{m=0}^{2n+1} E_m. \quad (5.66)$$

**Proof.** The proof is similar to the corresponding stationary case in Lemma 3.5.  $\square$

## 6 Time-dependent algebro-geometric solutions

In the final section, we extend the results of section 4 from the stationary HS hierarchy to the time-dependent case. In particular, we obtain Riemann theta function representations for the Baker-Akhiezer function, the meromorphic function  $\phi$  and the algebro-geometric solutions for the HS hierarchy.

Let us first consider the asymptotic properties of  $\phi$  in the time-dependent case.

**Lemma 6.1** *Assume that (2.2), (5.3) and (5.4) hold. Let  $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty\pm}, P_0\}$ ,  $(x, t_r) \in \mathbb{R}^2$ . Then, we have*

$$\phi(P) \underset{\zeta \rightarrow 0}{=} -u_x + O(\zeta), \quad P \rightarrow P_{\infty\pm}, \quad \zeta = z^{-1}, \quad (6.1)$$

$$\phi(P) \underset{\zeta \rightarrow 0}{=} i a \left( \prod_{m=1}^{2n+1} E_m \right)^{1/2} f_{n+1}^{-1} \zeta + O(\zeta^2), \quad P \rightarrow P_0, \quad \zeta = z^{1/2}. \quad (6.2)$$

**Proof.** The proof is identical to the corresponding stationary case in Lemma 4.1.  $\square$

Next, we study the properties of Abel map, which dose not linearize the divisor  $\mathcal{D}_{\hat{\mu}_0(x, t_r)} \hat{\mu}(x, t_r)$  in the time-dependent HS hierarchy. This is a remarkable difference between CH, MCH, HS hierarchies and other integrable soliton equations such as KdV and AKNS hierarchies. For that purpose, we introduce some notations of symmetric functions.

Let us define

$$\begin{aligned} \mathcal{S}_{k+1} &= \{\underline{l} = (l_1, \dots, l_{k+1}) \in \mathbb{N}_0^{k+1} \mid l_1 < \dots < l_{k+1} \leq n\}, \quad k = 0, \dots, n, \\ \mathcal{T}_{k+1}^{(j)} &= \{\underline{l} = (l_1, \dots, l_{k+1}) \in \mathcal{S}_{k+1} \mid l_m \neq j\}, \quad k = 0, \dots, n-1, \quad j = 0, \dots, n. \end{aligned} \quad (6.3)$$

The symmetric functions are defined by

$$\Psi_0(\bar{\mu}) = 1, \quad \Psi_{k+1}(\bar{\mu}) = (-1)^{k+1} \sum_{\underline{l} \in \mathcal{S}_{k+1}} \mu_{l_1} \cdots \mu_{l_{k+1}}, \quad k = 0, \dots, n, \quad (6.4)$$

and

$$\begin{aligned} \Phi_0^{(j)}(\bar{\mu}) &= 1, \\ \Phi_{k+1}^{(j)}(\bar{\mu}) &= (-1)^{k+1} \sum_{\underline{l} \in \mathcal{T}_{k+1}^{(j)}} \mu_{l_1} \cdots \mu_{l_{k+1}}, \\ k &= 0, \dots, n-1, \quad j = 0, \dots, n, \end{aligned} \quad (6.5)$$

where  $\bar{\mu} = (\mu_0, \dots, \mu_n) \in \mathbb{C}^{n+1}$ . The properties of  $\Psi_{k+1}(\bar{\mu})$  and  $\Phi_{k+1}^{(j)}(\bar{\mu})$  can be found in Appendix E [15]. Here we freely use these relations.

Moreover, for the HS hierarchy we have <sup>2</sup>

$$\begin{aligned} \widehat{F}_{r+1}(\mu_j) &= \sum_{s=(r-n) \vee 0}^{r+1} \hat{c}_s(\underline{E}) \Phi_{r+1-s}^{(j)}(\bar{\mu}), \\ \widetilde{F}_{r+1}(\mu_j) &= \sum_{s=0}^{r+1} \tilde{c}_{r+1-s} \widehat{F}_s(\mu_j) = \sum_{k=0}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1,k}(\underline{E}) \Phi_k^{(j)}(\bar{\mu}), \quad r \in \mathbb{N}_0, \quad \tilde{c}_0 = 1, \end{aligned} \quad (6.6)$$

where

$$\tilde{d}_{r+1,k}(\underline{E}) = \sum_{s=0}^{r+1-k} \tilde{c}_{r+1-k-s} \hat{c}_s(\underline{E}) \quad k = 0, \dots, r+1 \wedge n+1. \quad (6.7)$$

---

<sup>2</sup>  $m \wedge n = \min\{m, n\}$ ,  $m \vee n = \max\{m, n\}$

**Theorem 6.2** Assume that  $\mathcal{K}_n$  is nonsingular and (2.2) holds. Suppose that  $\{\hat{\mu}_j\}_{j=0,\dots,n}$  satisfies the Dubrovin equations (5.56), (5.57) on  $\Omega_\mu$  and remain distinct and  $\tilde{F}_{r+1}(\mu_j) \neq 0$  for  $(x, t_r) \in \Omega_\mu$ , where  $\Omega_\mu \subseteq \mathbb{R}^2$  is open and connected. Introducing the associated divisor  $\mathcal{D}_{\hat{\mu}_0(x, t_r)\underline{\hat{\mu}}(x, t_r)}$ , then

$$\partial_x \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x, t_r)\underline{\hat{\mu}}(x, t_r)}) = -\frac{2a}{\Psi_{n+1}(\bar{\mu}(x, t_r))} \underline{\mathcal{C}}(1), \quad (x, t_r) \in \Omega_\mu, \quad (6.8)$$

$$\begin{aligned} \partial_{t_r} \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x, t_r)\underline{\hat{\mu}}(x, t_r)}) &= -\frac{2a}{\Psi_{n+1}(\bar{\mu}(x, t_r))} \\ &\times \left( \sum_{k=0}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1, k}(\underline{E}) \Psi_k(\bar{\mu}(x, t_r)) \right) \underline{\mathcal{C}}(1) \\ &+ 2a \left( \sum_{\ell=1 \vee (n+1-r)}^{n+1} \tilde{d}_{r+1, n+2-\ell}(\underline{E}) \underline{\mathcal{C}}(\ell) \right), \\ &\quad (x, t_r) \in \Omega_\mu. \end{aligned} \quad (6.9)$$

In particular, the Abel map dose not linearize the divisor  $\mathcal{D}_{\hat{\mu}_0(x, t_r)\underline{\hat{\mu}}(x, t_r)}$  on  $\Omega_\mu$ .

**Proof.** It suffices to prove (6.9), since the proofs of (6.8) has been given in the stationary context of Theorem 4.3. Let us first give a fundamental identity (E.17) [15], that is

$$\Phi_{k+1}^{(j)}(\bar{\mu}) = \mu_j \Phi_k^{(j)}(\bar{\mu}) + \Psi_{k+1}(\bar{\mu}), \quad k = 0, \dots, n, \quad j = 0, \dots, n. \quad (6.10)$$

Then, together with (6.6) and (4.16), we have

$$\begin{aligned} \frac{\tilde{F}_{r+1}(\mu_j)}{\mu_j} &= \mu_j^{-1} \sum_{m=0}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1, m}(\underline{E}) \Phi_m^{(j)}(\bar{\mu}) \\ &= \mu_j^{-1} \sum_{m=0}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1, m}(\underline{E}) \left( \mu_j \Phi_{m-1}^{(j)}(\bar{\mu}) + \Psi_m(\bar{\mu}) \right) \\ &= \sum_{m=1}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1, m}(\underline{E}) \Phi_{m-1}^{(j)}(\bar{\mu}) \\ &\quad - \sum_{m=0}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1, m}(\underline{E}) \Psi_m(\bar{\mu}) \frac{\Phi_n^{(j)}(\bar{\mu})}{\Psi_{n+1}(\bar{\mu})}. \end{aligned} \quad (6.11)$$



So, using (6.11), (5.57), (E.9), (E.25) and (E.26) [15], we obtain

$$\begin{aligned}
& \partial_{t_r} \left( \sum_{j=0}^n \int_{Q_0}^{\hat{\mu}_j} \underline{\omega} \right) = \sum_{j=0}^n \mu_{j,t_r} \sum_{k=1}^n \underline{c}(k) \frac{a \mu_j^{k-1}}{y(\hat{\mu}_j)} \\
& = 2a \sum_{j=0}^n \sum_{k=1}^n \underline{c}(k) \frac{\mu_j^{k-1}}{\prod_{\substack{l=0 \\ l \neq j}}^n (\mu_j - \mu_l)} \frac{\tilde{F}_{r+1}(\mu_j)}{\mu_j} \\
& = 2a \sum_{j=0}^n \sum_{k=1}^n \underline{c}(k) \frac{\mu_j^{k-1}}{\prod_{\substack{l=0 \\ l \neq j}}^n (\mu_j - \mu_l)} \left( - \sum_{m=0}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1,m}(\underline{E}) \Psi_m(\bar{\mu}) \frac{\Phi_n^{(j)}(\bar{\mu})}{\Psi_{n+1}(\bar{\mu})} \right. \\
& \quad \left. + \sum_{m=1}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1,m}(\underline{E}) \Phi_{m-1}^{(j)}(\bar{\mu}) \right) \\
& = -2a \sum_{m=0}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1,m}(\underline{E}) \frac{\Psi_m(\bar{\mu})}{\Psi_{n+1}(\bar{\mu})} \sum_{k=1}^n \sum_{j=0}^n \underline{c}(k) (U_{n+1}(\bar{\mu}))_{k,j} (U_{n+1}(\bar{\mu}))_{j,1}^{-1} \\
& \quad + 2a \sum_{m=1}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1,m}(\underline{E}) \sum_{k=1}^n \sum_{j=0}^n \underline{c}(k) (U_{n+1}(\bar{\mu}))_{k,j} (U_{n+1}(\bar{\mu}))_{j,n-m+2}^{-1} \\
& = -\frac{2a}{\Psi_{n+1}(\bar{\mu})} \sum_{m=0}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1,m}(\underline{E}) \Psi_m(\bar{\mu}) \underline{c}(1) \\
& \quad + 2a \sum_{m=1}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1,m}(\underline{E}) \underline{c}(n-m+2) \\
& = -\frac{2a}{\Psi_{n+1}(\bar{\mu})} \sum_{m=0}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1,m}(\underline{E}) \Psi_m(\bar{\mu}) \underline{c}(1) \\
& \quad + 2a \sum_{m=1 \vee (n+1-r)}^{n+1} \tilde{d}_{r+1,n+2-m}(\underline{E}) \underline{c}(m). \tag{6.12}
\end{aligned}$$

Therefore, we complete the proof of (6.9).  $\square$

The analogous results hold for the corresponding divisor  $\mathcal{D}_{\hat{\mu}(x,t_r)}$  associated with  $\phi(P, x, t_r)$ .

The following result is a special form of Theorem 6.2, which provides the constraint condition to linearize the divisor  $\mathcal{D}_{\hat{\mu}_0(x,t_r)\hat{\mu}(x,t_r)}$  associated with  $\phi(P, x, t_r)$ . We recall the definitions of  $\hat{\underline{B}}_{Q_0}$  and  $\hat{\underline{\beta}}_{Q_0}$  in (4.20) and (4.21).

**Theorem 6.3** Assume that (2.2) holds and the statements of  $\{\mu_j\}_{j=0,\dots,n}$  in Theorem 6.2 are true. Then,

$$\partial_x \sum_{j=0}^n \int_{Q_0}^{\hat{\mu}_j(x,t_r)} \eta_1 = -\frac{2a}{\Psi_{n+1}(\bar{\mu}(x,t_r))}, \quad (x,t_r) \in \Omega_\mu, \quad (6.13)$$

$$\partial_x \hat{\beta}(\mathcal{D}_{\hat{\mu}(x,t_r)}) = \begin{cases} 2a, & n=1, \\ 2a(0,\dots,0,1), & n \geq 2, \end{cases} \quad (x,t_r) \in \Omega_\mu, \quad (6.14)$$

$$\begin{aligned} \partial_{t_r} \sum_{j=0}^n \int_{Q_0}^{\hat{\mu}_j(x,t_r)} \eta_1 &= -\frac{2a}{\Psi_{n+1}(\bar{\mu}(x,t_r))} \sum_{k=0}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1,k}(\underline{E}) \Psi_k(\bar{\mu}(x,t_r)) \\ &+ 2a \tilde{d}_{r+1,n+1}(\underline{E}) \delta_{n+1,r+1 \wedge n+1}, \quad (x,t_r) \in \Omega_\mu, \end{aligned} \quad (6.15)$$

$$\begin{aligned} &\partial_{t_r} \hat{\beta}(\mathcal{D}_{\hat{\mu}(x,t_r)}) \\ &= 2a \left( \sum_{s=0}^{r+1} \tilde{c}_{r+1-s} \hat{c}_{s+1-n}(\underline{E}), \dots, \sum_{s=0}^{r+1} \tilde{c}_{r+1-s} \hat{c}_{s+1}(\underline{E}), \sum_{s=0}^{r+1} \tilde{c}_{r+1-s} \hat{c}_s(\underline{E}) \right), \\ &\hat{c}_{-l}(\underline{E}) = 0, \quad l \in \mathbb{N}, \quad (x,t_r) \in \Omega_\mu. \end{aligned} \quad (6.16)$$

**Proof.** Equations (6.13) and (6.14) have been proved in the stationary case in Theorem 4.4. Equations (6.15) and (6.16) follows from (6.12), taking account into (E.9) [15].  $\square$ .

Motivated by Theorem 6.2 and Theorem 6.3, the change of variables

$$x \mapsto \tilde{x} = \int^x dx' \left( \frac{2a}{\Psi_{n+1}(\bar{\mu}(x',t_r))} \right) \quad (6.17)$$

and

$$\begin{aligned} t_r \mapsto \tilde{t}_r &= \int^{t_r} ds \left( \frac{2a}{\Psi_{n+1}(\bar{\mu}(x,s))} \sum_{k=0}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1,k}(\underline{E}) \Psi_k(\bar{\mu}(x,s)) \right. \\ &\quad \left. - 2a \sum_{\ell=1 \vee (n+1-r)}^{n+1} \tilde{d}_{r+1,n+2-\ell}(\underline{E}) \frac{\underline{c}(\ell)}{\underline{c}(1)} \right) \end{aligned} \quad (6.18)$$

linearizes the Abel map  $\underline{A}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(\tilde{x},\tilde{t}_r)} \hat{\mu}(\tilde{x},\tilde{t}_r))$ ,  $\tilde{\mu}_j(\tilde{x},\tilde{t}_r) = \mu_j(x,t_r)$ ,  $j=0,\dots,n$ . The intricate relation between the variables  $(x,t_r)$  and  $(\tilde{x},\tilde{t}_r)$  is detailedly studied in Theorem 6.4.

Next we shall provide an explicit representations of  $\phi$  and  $u$  in terms of the Riemann theta function associated with  $\mathcal{K}_n$ , assuming the affine part of  $\mathcal{K}_n$  to be nonsingular. Since the Abel map fails to linearize the divisor  $\mathcal{D}_{\hat{\mu}_0(x,t_r)\hat{\underline{\mu}}(x,t_r)}$ , one could argue that it suffices to consider the Dubrovin equations (5.56)-(5.57) and reconstruct  $u$  from the trace formula (5.66). By (4.24)-(4.32), one of the principal results reads as follows.

**Theorem 6.4** *Suppose that the curve  $\mathcal{K}_n$  is nonsingular, (2.2), (5.3) and (5.4) hold on  $\Omega$ . Let  $P = (z, y) \in \mathcal{K}_n \setminus \{P_0\}$ , and  $(x, t_r), (x_0, t_{0,r}) \in \Omega$ , where  $\Omega \subseteq \mathbb{R}^2$  is open and connected. Moreover, suppose that  $\mathcal{D}_{\hat{\underline{\mu}}(x,t_r)}$ , or  $\mathcal{D}_{\hat{\underline{\mu}}(x,t_r)}$  is nonspecial for  $(x, t_r) \in \Omega$ . Then,  $\phi$  and  $u$  have the following representations*

$$\begin{aligned} \phi(P, x, t_r) &= ia \left( \prod_{m=1}^{2n+1} E_m \right)^{1/2} f_{n+1}^{-1} \frac{\theta(\underline{z}(P, \hat{\underline{\mu}}(x, t_r))) \theta(\underline{z}(P_0, \hat{\underline{\mu}}(x, t_r)))}{\theta(\underline{z}(P_0, \hat{\underline{\mu}}(x, t_r))) \theta(\underline{z}(P, \hat{\underline{\mu}}(x, t_r)))} \\ &\quad \times \exp \left( d_0 - \int_{Q_0}^P \omega_{\hat{\mu}_0(x,t_r)P_0}^{(3)} \right), \end{aligned} \quad (6.19)$$

$$\begin{aligned} u(x, t_r) &= -\frac{1}{2} \sum_{m=0}^{2n+1} E_m + \frac{1}{2} \sum_{j=1}^n \lambda_j \\ &\quad - \frac{1}{2} \sum_{j=1}^n U_j \partial_{\omega_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(x, t_r)) + \underline{\omega})}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(x, t_r)) + \underline{\omega})} \right) \Big|_{\underline{\omega}=0} \end{aligned} \quad (6.20)$$

Moreover, let  $\mu_j$ ,  $j = 0, \dots, n$ , be nonvanishing on  $\Omega$ . Then, we have the following constraint

$$\begin{aligned} &2a(x - x_0) + 2a(t_r - t_{0,r}) \sum_{s=0}^{r+1} \tilde{c}_{r+1-s} \hat{c}_s(\underline{E}) \\ &= \left( -2a \int_{x_0}^x \frac{dx'}{\prod_{k=0}^n \mu_k(x', t_r)} - 2a \sum_{k=0}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1,k}(\underline{E}) \int_{t_{0,r}}^{t_r} \frac{\Psi_k(\bar{\mu}(x_0, s))}{\Psi_{n+1}(\bar{\mu}(x_0, s))} ds \right) \\ &\quad \times \sum_{j=1}^n \left( \int_{a_j} \tilde{\omega}_{P_{\infty+} P_{\infty-}}^{(3)} \right) c_j(1) \\ &\quad + 2a(t_r - t_{0,r}) \sum_{\ell=1 \vee (n+1-r)}^{n+1} \tilde{d}_{r+1,n+2-\ell}(\underline{E}) \sum_{j=1}^n \left( \int_{a_j} \tilde{\omega}_{P_{\infty+} P_{\infty-}}^{(3)} \right) c_j(\ell) \\ &\quad + \ln \left( \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(x, t_r))) \theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(x_0, t_{0,r})))}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(x, t_r))) \theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(x_0, t_{0,r})))} \right), \end{aligned} \quad (6.21)$$

$$(x, t_r), (x_0, t_{0,r}) \in \Omega$$

with

$$\begin{aligned} & \hat{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x, t_r)} \hat{\mu}(x, t_r)) \\ &= \hat{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x_0, t_r)} \hat{\mu}(x_0, t_r)) - 2a \left( \int_{x_0}^x \frac{dx'}{\Psi_{n+1}(\bar{\mu}(x', t_r))} \right) \underline{c}(1) \quad (6.22) \end{aligned}$$

$$\begin{aligned} &= \hat{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x, t_{0,r})} \hat{\mu}(x, t_{0,r})) \\ &\quad - 2a \left( \sum_{k=0}^{(r+1) \wedge (n+1)} \tilde{d}_{r+1,k}(\underline{E}) \int_{t_{0,r}}^{t_r} \frac{\Psi_k(\bar{\mu}(x, s))}{\Psi_{n+1}(\bar{\mu}(x, s))} ds \right) \underline{c}(1) \\ &\quad + 2a(t_r - t_{0,r}) \left( \sum_{\ell=1 \vee (n+1-r)}^{n+1} \tilde{d}_{r+1, n+2-\ell}(\underline{E}) \underline{c}(\ell) \right), \quad (6.23) \end{aligned}$$

$$(x, t_r), (x_0, t_{0,r}) \in \Omega.$$

**Proof.** Let us first assume that  $\mu_j, j = 0, \dots, n$ , are distinct and nonvanishing on  $\tilde{\Omega}$  and  $\tilde{F}_{r+1}(\mu_j) \neq 0$  on  $\tilde{\Omega}, j = 0, \dots, n$ , where  $\tilde{\Omega} \subseteq \Omega$ . Then, the representation (6.19) for  $\phi$  on  $\tilde{\Omega}$  follows by combining (5.28), (6.1), (6.2) and Theorem A.26 [15]. The representation (6.20) for  $u$  on  $\tilde{\Omega}$  follows from the trace formulas (5.66) and (F.89) [15]. In fact, since the proofs of (6.19) and (6.20) are identical to the corresponding stationary results in Theorem 4.5, which can be extended line by line to the time-dependent setting, here we omit the corresponding details. The constraint (6.21) then holds on  $\tilde{\Omega}$  by combining (6.13)-(6.16), and (F.88) [15]. Equations (6.22) and (6.23) is clear from (6.8) and (6.9). The extension of all results from  $(x, t_r) \in \tilde{\Omega}$  to  $(x, t_r) \in \Omega$  then simply follows by the continuity of  $\hat{\alpha}_{Q_0}$  and the hypothesis of  $\mathcal{D}_{\hat{\mu}(x, t_r)}$  being nonspecial for  $(x, t_r) \in \Omega$ .  $\square$

**Remark 6.5** A closer look at Theorem 6.4 shows that (6.22) and (6.23) equal to

$$\hat{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x, t_r)} \hat{\mu}(x, t_r)) = \hat{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x_0, t_r)} \hat{\mu}(x_0, t_r)) - \underline{c}(1)(\tilde{x} - \tilde{x}_0) \quad (6.24)$$

$$= \hat{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}_0(x, t_{0,r})} \hat{\mu}(x, t_{0,r})) - \underline{c}(1)(\tilde{t}_r - \tilde{t}_{0,r}), \quad (6.25)$$

based on the changing of variables  $x \mapsto \tilde{x}$  and  $t_r \mapsto \tilde{t}_r$  in (6.17) and (6.18). Hence, the Abel map linearizes the divisor  $\mathcal{D}_{\hat{\mu}_0(x, t_r)} \hat{\mu}(x, t_r)$  on  $\Omega$  with respect to  $\tilde{x}, \tilde{t}_r$ . This fact reveals that the Abel map does not effect the linearization of the divisor  $\mathcal{D}_{\hat{\mu}_0(x, t_r)} \hat{\mu}(x, t_r)$  in the time-dependent HS case.

**Remark 6.6** *Remark 4.8 is applicable to the present time-dependent context. Moreover, in order to obtain the theta function representation of  $\psi_j$ ,  $j = 1, 2, \dots$ , one can write  $\tilde{F}_{r+1}$  in terms of  $\Psi_k(\bar{\mu})$  and use (5.46), in analogy to the stationary case studied in Remark 4.9. Here we skip the corresponding details.*

Let us end this section by providing another principle result about algebro-geometric initial value problem of HS hierarchy. We will show that the solvability of the Dubrovin equations (5.56) and (5.57) on  $\Omega_\mu \subseteq \mathbb{R}^2$  in fact implies (5.3) and (5.4) on  $\Omega_\mu$ . As pointed out in Remark 4.12, this amounts to solving the time-dependent algebro-geometric initial value problem (5.1) and (5.2) on  $\Omega_\mu$ . Recalling definition of  $\tilde{F}_{r+1}(\mu_j)$  introduced in (6.6), then we may present the following result.

**Theorem 6.7** *Assume that (2.2) holds and  $\{\hat{\mu}_j\}_{j=0,\dots,n}$  satisfies the Dubrovin equations (5.56) and (5.57) on  $\Omega_\mu$  and remain distinct and nonzero for  $(x, t_r) \in \Omega_\mu$ , where  $\Omega_\mu \subseteq \mathbb{R}^2$  is open and connected. Moreover, suppose that  $\tilde{F}_{r+1}(\mu_j)$  in (5.57) expressed in terms of  $\mu_k$ ,  $k = 0, \dots, n$  by (6.6). Then  $u \in C^\infty(\Omega_\mu)$  defined by*

$$u = -\frac{1}{2} \sum_{m=0}^{2n+1} E_m + \frac{1}{2} \sum_{j=0}^n \mu_j, \quad (6.26)$$

*satisfies the  $r$ th HS equation (5.1), that is,*

$$\text{HS}_r(u) = 0 \quad \text{on } \Omega_\mu, \quad (6.27)$$

*with initial values satisfying the  $n$ th stationary HS equation (5.2).*

**Proof.** Given the solutions  $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in C^\infty(\Omega_\mu, \mathcal{K}_n)$ ,  $j = 0, \dots, n$  of (5.56) and (5.57), we introduce polynomials  $F_{n+1}$ ,  $G_n$ , and  $H_n$  on  $\Omega_\mu$ , which are exactly the same as in Theorem 4.10 in the stationary case

$$F_{n+1}(z) = \prod_{j=0}^n (z - \mu_j), \quad (6.28)$$

$$G_n(z) = \frac{1}{2} F_{n+1,x}(z), \quad (6.29)$$

$$z G_{n,x}(z) = -H_n(z) - u_{xx} F_{n+1}(z), \quad (6.30)$$

$$H_{n,x}(z) = 2u_{xx} G_n(z), \quad (6.31)$$

$$R_{2n+2}(z) = z^2 G_n^2(z) + z F_{n+1}(z) H_n(z), \quad (6.32)$$

where  $t_r$  is treated as a parameter. Hence let us focus on the proof of (5.1). Let us denote the polynomial  $\tilde{G}_r$  and  $\tilde{H}_r$  of degree  $r$  by

$$\tilde{G}_r(z) = \frac{1}{2}\tilde{F}_{r+1,x}(z) \quad \text{on } \mathbb{C} \times \Omega_\mu, \quad (6.33)$$

$$\tilde{H}_r(z) = -z\tilde{G}_{r,x}(z) - u_{xx}\tilde{F}_{r+1}(z) \quad \text{on } \mathbb{C} \times \Omega_\mu, \quad (6.34)$$

respectively. Next we want to establish

$$F_{n+1,t_r}(z) = 2(G_n(z)\tilde{F}_{r+1}(z) - F_{n+1}(z)\tilde{G}_r(z)) \quad \text{on } \mathbb{C} \times \Omega_\mu. \quad (6.35)$$

One computes from (5.56) and (5.57) that

$$F_{n+1,x}(z) = -F_{n+1}(z) \sum_{j=0}^n \mu_{j,x}(z - \mu_j)^{-1}, \quad (6.36)$$

$$F_{n+1,t_r}(z) = -F_{n+1}(z) \sum_{j=0}^n \tilde{F}_{r+1}(\mu_j) \mu_{j,x}(z - \mu_j)^{-1}. \quad (6.37)$$

Using (6.29) and (6.33) one concludes that (6.35) is equivalent to

$$\tilde{F}_{r+1,x}(z) = \sum_{j=0}^n (\tilde{F}_{r+1}(\mu_j) - \tilde{F}_{r+1}(z)) \mu_{j,x}(z - \mu_j)^{-1}. \quad (6.38)$$

Equation (6.38) has been proved in Lemma F.9 [15]. Hence this in turn proves (6.35).

Next, differentiating (6.29) with respect to  $t_r$  yields

$$F_{n+1,xt_r} = 2G_{n,t_r}. \quad (6.39)$$

On the other hand, the derivative of (6.35) with respect to  $x$ , taking account into (6.29), (6.30) and (6.33), we obtain

$$\begin{aligned} F_{n+1,t_rx} &= -2z^{-1}H_n\tilde{F}_{r+1} - 2z^{-1}u_{xx}F_{n+1}\tilde{F}_{r+1} + 2G_n\tilde{F}_{r+1,x} \\ &\quad - 2\tilde{G}_{r,x}F_{n+1} - 4\tilde{G}_rG_n. \end{aligned} \quad (6.40)$$

Combining (6.34), (6.39) and (6.40) we conclude

$$zG_{n,t_r}(z) = \tilde{H}_r(z)F_{n+1}(z) - H_n(z)\tilde{F}_{r+1}(z) \quad \text{on } \mathbb{C} \times \Omega_\mu. \quad (6.41)$$

Next, differentiating (6.32) with respect to  $t_r$ , inserting the expressions (6.35) and (6.41) for  $F_{n+1,t_r}$  and  $G_{n,t_r}$ , respectively, we obtain

$$H_{n,t_r}(z) = 2(H_n(z)\tilde{G}_r(z) - G_n(z)\tilde{H}_r(z)) \quad \text{on } \mathbb{C} \times \Omega_\mu. \quad (6.42)$$

Finally, taking the derivative of (6.41) with respect to  $x$  and inserting (6.29), (6.31) and (6.33) for  $F_{n+1,x}$ ,  $H_{n,x}$  and  $\tilde{F}_{r+1,x}$ , respectively, yields

$$zG_{n,t_r x} = F_{n+1}\tilde{H}_{r,x} + 2G_n\tilde{H}_r - 2u_{xx}G_n\tilde{F}_{r+1} - 2H_n\tilde{G}_r. \quad (6.43)$$

On the other hand, differentiating (6.30) with respect to  $t_r$ , using (6.35) and (6.42) for  $F_{n+1,t_r}$  and  $H_{n,t_r}$ , respectively, leads to

$$zG_{n,xt_r} = 2G_n\tilde{H}_r - 2H_n\tilde{G}_r - u_{xxt_r}F_{n+1} - 2u_{xx}(G_n\tilde{F}_{r+1} - \tilde{G}_rF_{n+1}) \quad (6.44)$$

Hence, combining (6.43) and (6.44) then yields

$$-u_{xxt_r} - \tilde{H}_{r,x} + 2u_{xx}\tilde{G}_r = 0. \quad (6.45)$$

Thus we proved (5.12)-(5.17) and (5.38)-(5.40) on  $\mathbb{C} \times \Omega_\mu$  and hence conclude that  $u$  satisfies the  $r$ th HS equation (5.1) with initial values satisfying the  $n$ th stationary HS equation (5.2) on  $\mathbb{C} \times \Omega_\mu$ .  $\square$

**Remark 6.8** *The result in Theorem 6.7 is presented in terms of  $u$  and  $\{\mu_j\}_{j=0,\dots,n}$ , but of course one can provide the analogous result in terms of  $u$  and  $\{\nu_l\}_{l=1,\dots,n}$ .*

The analog of Remark 4.13 directly extends to the current time-dependent HS hierarchy.

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